

Final Homework — Auction Theory (96573)

Note: No extensions will be given. I advise to start working on this as early as possible.

Question 1 (30 points)

In this question we will examine the revenue equivalence theorem, and its implications, from a different perspective. Suppose we have risk-neutral players, with private values that are drawn i.i.d. from some cumulative distribution F . We will consider a family \mathcal{F} of auctions in which each player bids a number, and the player with the highest bid wins. (for example first-price and second-price are both such auctions).

Fix any such auction, A , with symmetric and non-decreasing equilibrium strategies $b(x)$. Let $m_A(z)$ denote the expected payment of a player that bids $b(z)$, assuming all other players play according to $b(\cdot)$ (the expectation is taken over the values of the other players). Let $G_i(y) = Pr(\max_{j \neq i} v_j \leq y)$, i.e. the cumulative distribution to the highest value besides v_i . Since all values are drawn from the same distribution F , all the G_i 's are identical, and we will denote $G_i(y) = G(y)$ and $g(y) = G'(y)$.

1. Show that the fact that $b(\cdot)$ is an equilibrium implies that $m'_A(z) = g(z) \cdot z$. Guidance: start by writing a formula to the expected utility, $u_A(x, z)$, for a player with value x that bids $b(z)$. Then, the equilibrium property implies that $u_A(x, z)$ is maximized for $z = x$, and this will give you the result. Explain in details your answer.
2. Assume that the expected payment of a player with value zero is zero. Based on your previous answer, what is the expected payment of a player with positive value? Prove your answer.
3. In an all-pay auction, each player submits a bid, the player with the highest bid wins, and every player pays her bid (both the winner and the losers). Notice that an all-pay auction belongs to the family \mathcal{F} . What will be the equilibrium strategy $b(x)$ for this auction? Hint: remember that a player always pays her bid.
4. Suppose that the probability distribution F is the uniform distribution on $[0, 1]$. What will be the function $b(x)$ from the previous question? Prove directly that this function $b(\cdot)$ is indeed an equilibrium, and compute the expected revenue of the all-pay auction for this equilibrium. Is it higher or lower than the revenue of the first-price auction, that we computed in class? Compare this analysis and conclusion to the revenue equilibrium theorem.

Question 2 (30 points)

In this question we will focus on “team auctions”: an auctioneer needs to hire one of two teams, Q and T , to perform a certain job. Each team is composed of several different players, and the two teams are disjoint. Each player, i , incurs a cost $c_i \geq 0$ if her team is chosen to perform the job, and this cost is private to each player. We assume here that the auctioneer must ensure a non-negative utility to the players, otherwise the players will not agree to participate, and thus the auctioneer needs to pay the players. The utility of each player is her payment minus her cost.

The auctioneer wants to design a truthful auction, and to pay as little as possible. Specifically, let $M_A(c)$ denote the total payment of the auctioneer when the players report the costs $c = (c_1, \dots, c_n)$, and define the “frugality ratio” of an auction A as,

$$FR(A) = \sup_c \frac{M_A(c)}{\max\{\sum_{i \in Q} c_i, \sum_{i \in T} c_i\}}$$

(i.e. the ratio between the actual payment and the second-lowest cost of any one of the teams). We search for an auction with a small frugality ratio, which means that we pay not much more than the second-lowest cost. We will see that VCG is almost the best possible auction if $|Q| = |T| = x$.

1. Describe a **truthful** auction with frugality ratio 1 for the case that $x = 1$ (i.e. each team contains one player).
2. Describe the format of the VCG auction for disjoint teams of size x . Prove that $FR(VCG) = x$, i.e. prove that $FR(VCG) \leq x$ and show an example for which $\frac{M_A(c)}{\max\{\sum_{i \in Q} c_i, \sum_{i \in T} c_i\}} = x$.
3. We now consider any other truthful deterministic auction, A . For any player $i \in Q$, let c_Q^i be a cost vector for the players of team Q , where $c_j = 0$ for every $j \in Q, j \neq i$, and $c_i = 3$. Similarly define c_T^i for every $i \in T$. Fix two players $i \in Q$ and $j \in T$, suppose that the bids are (c_Q^i, c_T^j) , and that Q wins. Prove that then, if all players in Q bid 0 and players in T still bid c_T^j then (i) Q continues to win, and (ii) the payment to player i is at least 3. Hint: use the notions of value monotonicity and threshold value from the second homework.
4. For a player $i \in Q$, let l_i denote the number of players $j \in T$ such that T wins when the bids are (c_Q^i, c_T^j) . Similarly define l_i for any $i \in T$. Prove that there exists a player i (either in T or in Q) such that $l_i \geq x/2$.
5. Use the above to show that the frugality ratio of A is at least $x/2$, when $|Q| = |T| = x$. Hint: calculate the ratio $\frac{M_A(c)}{\max\{\sum_{i \in Q} c_i, \sum_{i \in T} c_i\}}$ for cases that are similar to what is described in section 3.

Question 3 (40 points)

This question is based on the paper “Late and Multiple Bidding in Second-Price Internet Auctions: Theory and Evidence Concerning Different Rules for Ending an Auction”, by Ockenfels and Roth, published in *Games and Economic Behavior*, volume 55, pages 297-320. A link to the paper is given in the course website.

1. Describe the phenomena and the open question that this paper aims to solve. What is the main answer?
2. Give a survey of the results in the paper. This should be comprehensive, but short and to the point. Describe both the theoretical results and the empirical results.
3. Choose one theorem from the paper and redo it: state the model and assumptions, the theorem itself, and the proof. Make sure you write it in a self-contained way, so that a person that never read the paper can read and understand your answer.
4. Describe an alternative to the main answer that the paper provides (still assuming that eBay bidders are rational), and justify your answer.