K-PRICE AUCTIONS: REVENUE INEQUALITIES, UTILITY EQUIVALENCE, AND COMPETITION IN AUCTION DESIGN†

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July, 2003

Abstract. This paper discusses revenue inequalities, utility equivalence, and a model of competition in auction design in symmetric equilibrium of k-price auctions, $k \geq 1$, all in the setup of symmetric independent-private-value auctions. Our recommendation to organizers of auctions is to conduct k-price auctions, $k \geq 3$ in environments in which buyers are risk-seeking. The recommendation is given both for, a setup in which each organizer is a monopolist, and for one of oligopolistic competition.

1.Introduction. Here we discuss the relevant literature and describe our main results. Every definition and theorem depends on particular assumptions. For simplicity we do not describe these (relatively common) assumptions in the introduction. A reader who wishes to quote our theorems should read the precise

†First version (titled "Internet Auctions"): October 1997. This version was split to Monderer and Tennenholtz (2000), and to the manuscript: "Internet Auctions: Are They Gamblers' Attraction?". The current version is a newer version of the later one with a new title. We are thankful for useful comments from the participants of the CS-economics seminar at Stanford University, the participants of the economics seminars at Tel-Aviv University, Mannheim University, and the University of Bonn, and the participants of the computer science seminar at the Weizmann Institute. We would also like to thank Haim Reisman, Yoav Shoham, Doron Sonsino, Hal Varian, and specially Shlomit Hon-Snir for helpful comments and discussions. Part of this research was supported by the Technion Fund for the Promotion of Research.

assumptions given in the body of the paper. However, there are three basic characteristics that we assume throughout the whole paper:

- ⋄ Symmetric independent-private-value model.
- ⋄ Symmetric equilibrium.
- ⋄ No reserve prices.

This paper deals with k-price auctions, $k \geq 1$. As far as we know k-price auctions, for $k \geq 3$, are not used in real life. Consequently, such auctions have not been extensively analyzed in the literature. Equilibrium in third-price auctions was discussed in Kagel and Levin (1993). k-price auctions, $k \geq 3$, were discussed in Wolfstteter (1996), Monderer and Tennenholtz (2000), and Tauman (2001). In the current paper we use a normative approach. We recommend to sellers in an environment in which buyers are risk-seeking (or fun-seeking)[1] to take advantage of their special type of clients by using high degree auctions. It is well-known that a risk-neutral agent can sell a lottery ticket with a negative expected payoff to a risk-seeking agent. However, gambling is an illegal activity in certain countries. Auctions are legal, and can be considered a substitute to lottery, with the uncertainty regarding the bids of the other agents serving as an implicit lottery mechanism. The question is, which auctions yield high revenue.

It is shown first that third-price auctions exploit the love for gambling of risk-seeking agents better than second-price auctions[2]. More precisely, we show that when facing risk-seeking buyers, the revenue of a seller in a third-price auction is higher than in a second-price auction. By Holt (1980), Riley and Samuelson (1981) and Maskin and Riley (1984) (see also Krishna (2002)), second-price auctions yield higher expected revenue than first-price auctions. That is,

- $R_3 \geq R_2 \geq R_1$ for risk-seeking buyers.

As a byproduct we show:

- $R_3 \leq R_2 \leq R_1$ for risk-averse buyers,

and strict inequalities hold for strict preference relations.

The rest of the paper deals with buyers with constant absolute risk attitude, that is, their utility function has the form $u(x) = \frac{e^{\lambda x}-1}{\lambda}$. A buyer with such a

[1] In the sense that they have convex utility functions.

[2] In a third-price auction every bid below the valuation is weakly dominated. Moreover, Monderer and Tennenholtz (2000) proved that buyers overbid in equilibrium of any third-price auction regardless of their risk attitude (they may be risk averse, risk seeking or have any alternating attitude to risk). This is a generalization of Kagel and Levin (1993), who proved this result for risk-neutral buyers and for buyers with constant absolute risk aversion. On the intuitive level, this overbidding property reflects the gambling nature of third-price auction.

utility function has a constant absolute risk-seeking (CARS) if $\lambda > 0$, and constant absolute risk-aversion (CARA) if $\lambda < 0$. We show:

• $R_{k+1} \geq R_k$ for buyers with CARS, and $R_k \geq R_{k+1}$, for buyers with CARA.

Most of the research in Auction Theory focuses on the seller's perspective. The Optimal Auction Theorem (Myerson (1981)), which characterizes auction mechanisms that maximize the seller's revenue, and the Revenue Equivalence Principle[3], which provides conditions under which a seller is indifferent between various auctions are well-known examples. When following Myerson's proof of the Revenue Equivalence Principle, it can be seen that it follows from a Utility Equivalence Principle for risk-neutral buyers, and that these two principles are equivalent. That is, the seller is indifferent between two auction mechanisms if and only if every potential buyer is indifferent between them.

Matthews (1983, 1987) were the first attempts to compare auction mechanisms from the buyers' point of view, when the buyers were not risk-neutral. Matthews (1983) showed that when a buyer has constant absolute risk attitude, she is indifferent between first- and second-price auctions, that is, $B_1(v) = B_2(v)$ for every type v. This theorem is generalized here to all k-price auctions. We show:

• For buyers with constant absolute risk attitude, $B_k(v) = B_{k+1}(v)$ for every type v, provided that all other parameters (the number of buyers, and the distribution of types) are kept fixed.

Hon-Snir (2002) gave a general utility equivalence principle, that holds for buyers with constant absolute risk attitude. Furthermore, she showed that this equivalence principle holds if and only if the buyers have constant absolute risk-attitude. Hon-Snir's theorem does not require a continuous or symmetric setup. Using Hon-Snir's result has significantly simplified our original proof of the utility equivalence in k-price auctions.

Consider the following model of competition introduced by McAfee (1993): An *auction-selection game* is the following two-stage Bayesian game: There are m sellers and n buyers, $n \geq m \geq 2$. At Stage 1, each seller chooses an auction mechanism out of a class of k-price auctions, $k \in \{1, 2, ..., K\}$, $K \leq n$. At stage 2, each buyer receives his type and proceeds to choose an auction place and to submit a bid. All auctions are conducted simultaneously.[4]

[3] Vickrey (1961), Ortega-Reichert (1968), Holt (1980), Harris and Raviv (1981), Myerson(1981), Riley and Samuelson (1981).

[4] In particular, a buyer can not move to another auction if she does not win. See Peters and Severinov (2001) for a a model that allows buyers to switch to another auction if outbid.

McAfee (1993) gave an example of an auction selection game, in which equilibrium does not exist. He analyzed the auction selection game, using a weaker concept of equilibrium. In this equilibrium sellers ignore the impact that changes in auctions have on the payoffs associated with the equilibrium in the buyers' subgame. Peters and Severinov (1997) discussed another weak concept of equilibrium of auction selection games in a setup with infinite number of buyers and sellers. Burguet and Sákovics (1999) analyzed an auction selection game in which the set of available auctions for the sellers contains only second-price auctions with different reserve prices. All the above mentioned papers deal with risk-neutral buyers. They all mention that auction selection games seem to be analytically intractable. However, in our setup we prove:

- When buyers have constant absolute risk-attitude, or they are risk-neutral, every buyer picks every auction location with the same probability ($\frac{1}{m}$), at every buyer-symmetric subgame perfect equilibrium of the auction selection game,
- When the buyers have constant absolute risk-aversion, every seller chooses to conduct a first-price auction, at every buyer-symmetric subgame perfect equilibrium of the auction selection game,
- When the buyers have constant absolute risk-seeking, every seller chooses to conduct a K-price auction, at every buyer-symmetric subgame perfect equilibrium of the auction selection game,
- For risk-neutral buyers, every selection of auctions by the sellers is sustainable by a buyer-symmetric subgame perfect equilibrium of the auction selection game.

The intuition is as follows: Because of the utility equivalence principle mentioned above, a buyer's preferences over k-price auctions depend only on the number of participants. Hence, in equilibrium every auction location has the same number of expected participants. Given the buyers' behavior, every seller, independently of the other sellers' choices, has to choose out of a set of k-price auctions with the same distribution of participants. Therefore every seller acts in equilibrium as a monopolist, applying the expected revenue inequalities shown above.

We believe that many of the participants in internet auctions are risk-seeking much like casinos' clients. Hence, conducting k-price auctions may be a good choice for the sellers in internet auctions.[5]

[5] Unfortunately, however, our model of competition does not seem appropriate for the Internet,

Having made our recommendation, we would like to respond to an objection, normally raised by people who are not computer scientists: k-price auctions, $k \geq 2$, are sealed-bid auctions, and given environments in which sellers have low credibility (e.g., many of the internet auctions), it seems that buyers may fear cheating on the side of the organizers. That is, it seems beneficial for a seller, after observing the bids, to submit $k-1$ bids which are a little bit lower than the highest received bid. This way, she increases her revenue without risking the loss of a sale. Such a possibility may cause buyers to behave as if they participate in a first-price auction, and such a behavior may yield a lower revenue than the one realized in English auctions. To prevent such a phenomenon, cryptographic techniques can be used. For example, the buyers may be asked to broadcast encrypted bids (to be received by all buyers), and then in a later stage (after all bids have arrived) to decrypt their bids.[6] One may wish, though, to find public mechanisms (which do not use available cryptographic techniques) which are equivalent to k-price auction in the same sense that Dutch auction[7] and English auction (with private values) are equivalent to first- and second- price auctions respectively. Finding such mechanisms is an open problem for us.[8]

2. k-price Auctions. In this section we present the basic definitions needed for the analysis of k-price auctions of a single item in the independent-private-value model. We mainly follow Monderer and Tennenholtz (2000).

We assume that the seller is risk-neutral with zero valuation of the object, and that she sets a zero reserve price. There are n potential buyers of the object, to which we refer as *buyers*, denoted by $1, 2, \ldots, n$, $n \geq 2$. The set of buyers is denoted by N. The valuation v_i of Buyer i is drawn from the interval $[0, 1]$ according to a random variable $\tilde{v_i}$ which is distributed according to a distribution function G. For any subset of buyers $M \subseteq N$, we denote by P_G^M the product probability measure induced by G on $[0, 1]^M$. The expectation operator with respect to P_G^M is denoted

where buyers can move to another auction if outbid (see Peters and Severinov(2001) for a more appropriate model).

[6]See Linial (1992) for a thorough review of computer science techniques that have game-theoretic implications.

[7]As is common in the early auction literature we use the term "Dutch Auction" for a descending clock auction. Some internet auction sites use the term "Dutch auctions" to describe multi-unit auctions or versions of the English auction.

[8]The reader should notice that a recommendation to use k-price auctions, where k is **big** relative to the number of participants , may be problematic if it is not augumanted with a recommendation to use appropriate reservation prices. In the extreme example of n-price auctions with n participants, if the system is somewhat noisy, e.g., a buyer may not participate accidentally, the revenue may reduce to 0.

by E_G^M. When $M = \{i\}$, $E_G^{\{i\}}$ is denoted by E_G^i, and when $M = N \setminus \{i\}$, $E_G^{N\setminus\{i\}}$ is denoted by E_G^{-i}.

We assume that all buyers have the same Von-Neumann Morgenstern utility function $u(x)$, $-\infty < x < \infty$, which satisfies the following assumptions:

U1 $u(0) = 0$.

U2 u is twice continuously differentiable.

U3 $u'(x) > 0$ for every $x \in R$.

The buyers are risk-averse if $u'' \leq 0$. The set of all such functions u is denoted by RA. The buyers are strictly risk-averse, or $u \in SRA$, if $u'' < 0$. Similarly the buyers are risk-seeking, or $u \in RS$, if $u'' \geq 0$ and the buyers are strictly risk-seeking, or $u \in SRS$, if $u'' > 0$. The buyers are risk-neutral, or $u \in RN$, if $u'' = 0$ (that is $u(x) = ax$ for all x, for some $a > 0$). A buyer with a utility function u has *constant absolute risk-attitude* (or $u \in CAR$) if $\frac{u''}{u'}$ is a constant function. For every real number $\lambda \neq 0$ let $w_\lambda(x) = \frac{e^{\lambda x}-1}{\lambda}$. For $\lambda = 0$, let $w_0(x) = x$. It is well-known that $u \in CAR$ and $\frac{u''}{u'} = \lambda$ if and only if $u = cw_\lambda$, for some $c > 0$. Let $u = cw_\lambda \in CAR$. If $\lambda < 0$, we say that the buyer has *constant-absolute risk-aversion* ($u \in CARA$). If $\lambda > 0$, the buyer has *constant-absolute risk-seeking*($u \in CARS$). Obviously $CARA \subset SRA$, $CARS \subset SRS$.

In a k-price auction, $k \geq 1$, the winner (i.e., the buyer who gets the object) is the one with the highest bid. In a tie, the winner is determined by a lottery with equal probability for each participant with the maximal bid. The winner pays the reverse k-order statistics of the sequence of bids. That is, if the bids are ordered $x_{[1]} \geq x_{[2]} \cdots \geq x_{[n]}$, the winner pays $x_{[k]}$. If $n < k$ the winner pays 0.

For every $k \geq 1$, $n \geq \max\{2, k\}$, a utility function u which satisfies $U1 - U3$, and a distribution function G, we denote by $A(k, n, u, G)$ the k-price auction with these parameters. When some of the parameters are clear we may omit them.

A *strategy* (of any buyer) in the auction $A(k, n, u, G)$ is a bounded and Borel measurable function $b : [0, 1] \to [0, \infty)$. For a strategy b, we denote by $\hat{b}$ the profile of strategies $(b_i)_{i\in N}$, where $b_i = b$ for every $i \in N$. We use the symbol $-i$ as it is common in economics. For example, the sequence $(b_j)_{j\neq i}$ is denoted by $\hat{b}_{-i}$, for every $v = (v_i)_{i\in N} \in [0, 1]^N$, $v_{-i} = (v_j)_{j\in N\setminus\{i\}}$, and $\hat{b}_{-i}(v_{-i}) = (b_j(v_j))_{j\neq i}$. For every $n \geq k \geq 1$ and any sequence $(x_1, ..., x_n)$ of real numbers we denote the reverse k-order statistics of the sequence by $X_{[k]}(x_1, ..., x_n)$ or by $x_{[k]}$. Let b be a strategy. For every possible type of Buyer i, v_i, and for every possible bid x_i, denote by $E_i^k(x_i, b|v_i)$ the expected utility of buyer i in the k-price auction $A(k, n, u, G)$ given that his type is v_i, he bids x_i, and each other buyer uses the strategy b.[9] That is,

$$E_i^k(x_i, b|v_i) = E_G^{-i}\left(u(v_i - X_{[k]}(x_i, \hat{b}_{-i}(\tilde{v}_{-i})))\tau_i((x_i, \hat{b}_{-i}(\tilde{v}_{-i})))\right), \tag{2.1}$$

where, $\tau_i(y_1, ..., y_n)$ is the probability that i wins the object, when buyer j bids y_j for every j. That is, if $s(y_1, ..., y_n)$ denotes the number of maximal bids in $(y_1, ..., y_n)$, $\tau_i(y_1, ..., y_n) = \frac{1}{s(y_1,...,y_n)}$ if $y_i \geq \max(y_j)_{j=1}^n$, and $\tau_i(y_1, ..., y_n) = 0$ otherwise.

A strategy b is a *symmetric equilibrium* strategy in $A(k, n, u, F)$ if for every buyer i, and every $v_i \in [0, 1]$, $\max_{x_i \in [0,\infty)} E_i^k(x_i, b|v_i)$ is attained at $x_i = b(v_i)$. In Sections 2 and 3 we will deal only with regular auctions. We need a definition of a regular distribution function of a random variable that is distributed in $[0, 1]$. Such a distribution function F is called *regular* if it satisfies the following properties:

D0 $F(0) = 0$ and $F(1) = 1$.

D1 F is twice continuously differentiable in $[0, 1]$.

D2 $F'(x) > 0$ for every $x > 0$.

D3 $\lim_{x \to 0} \frac{F(x)}{F'(x)} = 0$.[10]

Note that $D3$ implies that the function which is $\frac{F(x)}{F'(x)}$ for $0 < x \leq 1$ and 0 at $x = 0$ is a continuous function on $[0, 1]$. We denote this function by $\frac{F(x)}{F'(x)}$. Hence, $\frac{F(0)}{F'(0)} = 0$.[11]

Definition: $A(k, n, u, F)$ is a *regular auction* if F is a regular distribution function, and $A(k, n, u, F)$ possesses a unique continuous and symmetric equilibrium strategy b, which satisfies:

(1) $b(0) = 0$.

(2) b is differentiable in $(0, 1]$.

(3) b is increasing in $[0, 1]$.

Lemma 2.1.

(2.1.1) *Let F be a regular distribution function. If $(\frac{u}{u'})' > 0$, $A(1, n, u, F)$ is a regular auction.*

[9] Technically, the conditional expectation function is not uniquely defined at a single type v_i. As it is commonly done, we pick the particular natural version of the conditional expectation given in (2.1).

[10] Note that in most of the auction literature D0, D1, D2, and $F'(0) > 0$ are assumed. In such a case assumption $D3$ is satisfied. However we make the weaker assumption $D3$ because it implies the following desired property: If F satisfies $D0 - D3$, so does F^n for every integer $n \geq 1$, where $F^n(x) = (F(x))^n$. By a repeated application of L'Hopital rule it can be verified that if F satisfies $D0 - D2$ and there exists an integer $n \geq 1$ for which the n^{th} derivative $F^{(n)}$ of F exists in $[0, 1]$ and $F^{(n)}(0) \neq 0$, then F satisfies $D3$.

[11] Actually all the results in this paper can be proved under weaker conditions than $D0 - D3$. These conditions allow F' to get the value ∞ at 0, e.g., $F(x) = \sqrt{x}$.

(2.1.2) *Let F be a regular distribution function, and let $k \geq 3$. If $A(k,n,u,F)$ possesses a continuous symmetric equilibrium strategy, then $A(k,n,u,F)$ is a regular auction.*

Proof. (2.1.1) follows from Maskin and Riley (1984). (2.1.2) follows from Monderer and Tennenholtz (2000).

3. Revenue Inequalities. Let $A(k,n,u,G)$ be a k-price auction. For every symmetric equilibrium strategy b we denote the expected revenue of the seller in equilibrium by $S(k,n,u,G,b)$. If $A(k,n,u,F)$ is a regular auction we denote by $S(k,n,u,F)$ the expected revenue in the unique continuous and symmetric equilibrium strategy. When n, F are fixed, we denote the expected revenue by S_k^u. It was proved in Holt (1980), Riley and Samuelson (1981), and in Maskin and Riley (1984) that for $u \in RA$, $S_1^u \geq S_2^u$. That is, the expected revenue of the seller in a first-price auction is not smaller than his expected revenue in a second-price auction, when the buyers are risk-averse. It is easily verified that the inequality is reversed when the buyers are risk-seeking, and that strict inequalities hold when the buyers have strict attitude to risk. Hence we have:

Theorem 3.1 (Holt, Maskin, Riley, Samuelson).
Let $n \geq 2$. Let $A(k,n,u,F)$ be regular auctions for $k = 1,2$.

(3.1.1) *if the buyers are risk averse (that is, $u \in RA$), then $S_1^u \geq S_2^u$.*
(3.1.2) *if the buyers are strictly risk-averse (that is, $u \in SRA$), then $S_1^u > S_2^u$.*
(3.1.3) *if the buyers are risk-seeking (that is, $u \in RS$), then $S_2^u \geq S_1^u$.*
(3.1.4) *if the buyers are strictly risk-seeking (that is, $u \in SRS$), then $S_2^u > S_1^u$.*

Theorem 3.2. *Let $n \geq 3$. Let $A(k,n,u,F)$ be regular auctions for $k = 2,3$.*

(3.2.1) *if the buyers are risk averse (that is, $u \in RA$),then $S_2^u \geq S_3^u$.*
(3.2.2) *if the buyers are strictly risk-averse (that is, $u \in SRA$), then $S_2^u > S_3^u$.*
(3.2.3) *if the buyers are risk-seeking (that is, $u \in RS$), then $S_3^u \geq S_2^u$.*
(3.2.4) *if the buyers are strictly risk-seeking (that is, $u \in SRS$), then $S_3^u > S_2^u$.*

In the proof of Theorem 3.2 we use the following "strict" version of Jensen's inequality:

Jensen Inequality. *Let b be an increasing continuous function on $[0,1]$.*

(1) *if $u \in SRS$, then for every $0 < x \leq 1$*

$$u\left(\int_0^x (x - b(t))F'(t)dt\right) < \int_0^x u(x - b(t))F'(t)dt.$$

(2) *if $u \in SRA$ then for every $0 < x \leq 1$*

$$u\left(\int_0^x (x - b(t))F'(t)dt\right) > \int_0^x u(x - b(t))F'(t)dt.$$

Proof. The assertion follows easily by an obvious modification of the proof of Theorem 3.3 in Rudin (1974) and the fact that $u(0) = 0$. □

We also need the following lemma, which will be used in the proof of other theorems as well.

Lemma 3.3. *For $n \geq L \geq 2$ denote by b_L the equilibrium strategy in the regular L-price auction $A(L, n, u, F)$. Define $S_L(v_1)$ to be the expected revenue of the seller from Buyer 1 in equilibrium, given that $\tilde{v}_1 = v_1$. Then*

$$S_L(v_1) = \Delta(n, L)\int_{t=0}^{v_1} b_L(t)F(t)^{n-L}\frac{(F(v_1) - F(t))^{L-2}}{(L-2)!}F'(t)dt, \tag{3.1}$$

where

$$\Delta(n, L) = (n-1)(n-2)\cdots(n-L+1).$$

Proof. Note that

$$S_L(v_1) = \Delta(n, L)\int_{v_2=0}^{v_1}\cdots\int_{v_L=0}^{v_{L-1}} b_L(v_L)\int_{v_{L+1}=0}^{v_L}\cdots\int_{v_n=0}^{v_L} dP_F^{\{2,\ldots,n\}}(v_2, \ldots, v_n),$$

Therefore,

$$S_L(v_1) = \Delta(n, L)\int_{v_2=0}^{v_1}\cdots\int_{v_L=0}^{v_{L-1}} b_L(v_L)F(v_L)^{n-L}dP_F^{\{2,\ldots,L\}}(v_2, \ldots, v_L), \tag{3.2}$$

Changing the order of integration in (3.2) yields (3.1) □

Proof of Theorem 3.2. We prove the inequality $S_3^u > S_2^u$ for strictly risk-seeking buyers. The other inequalities are similarly proved.

Let $b = b_3^u$ be the unique symmetric equilibrium strategy in $A(3, n, u, F)$. By Theorem *AT* in Monderer and Tennenholtz (2000),

$$\int_{t=0}^{x} u(x - b(t))F(t)^{n-3}F'(t)dt = 0, \qquad \text{for every } 0 \leq x \leq 1. \tag{3.3}$$

As u is strictly convex and $u(0) = 0$, applying Jensen's inequality to (3.3) yields:

$$u\left(\int_{t=0}^{x} (x - b(t))F(t)^{n-3}F'(t)dt\right) < \int_{t=0}^{x} u(x - b(t))F(t)^{n-k}F'(t)dt = 0. \tag{3.4}$$

Since u is increasing and $u(0) = 0$,

$$\int_{t=0}^{x} (x - b(t))F(t)^{n-3}F'(t)dt < 0.$$

Therefore,

$$\int_{t=0}^{x} xF(t)^{n-3}F'(t)dt < \int_{t=0}^{x} b(t)F(t)^{n-3}F'(t)dt. \tag{3.5}$$

As (3.5) holds for every $x \in [0, 1]$, for every $0 < y \le 1$

$$\int_{x=0}^{y} \left(\int_{t=0}^{x} xF(t)^{n-3}F'(t)dt \right) F'(x)dx < \int_{x=0}^{y} \left(\int_{t=0}^{x} b(t)F(t)^{n-3}F'(t)dt \right) F'(x)dx. \tag{3.6}$$

A direct computation of the internal integral of the left-hand-side of (3.6), and changing the order of integration of the right-hand-side of (3.6) yield

$$\frac{1}{n-2} \int_{x=0}^{y} xF(x)^{n-2}F'(x)dx < \int_{t=0}^{y} b(t)F(t)^{n-3}(F(y) - F(t))F'(t)dt. \tag{3.7}$$

By (3.1), multiplying the inequality (3.7) by $(n-1)(n-2)$, and renaming y as v_1 yields

$$S_2(v_1) < S_3(v_1), \quad \text{for every } 0 < v_1 \le 1.$$

By symmetry, $S_L^u = n \int_{v_1=0}^{1} S_L(v_1)F'(v_1)dv_1$for $L = 2, 3$. Therefore $S_2^u < S_3^u$. □

One would expect Theorem 3.2 to have an extension to general monotonicity in k. That is, if $u \in SRA$, $S_k^u > S_{k+1}^u$, etc. We could neither prove this general monotonicity result, nor find a counter example. We can however prove it for buyers with constant absolute risk attitude.

Theorem 3.4. *Let $k \ge 1$. Let $n \ge k + 1$. Let $u \in CAR$ and assume that $A(k, n, u, F)$ and $A(k + 1, n, u, F)$ are regular auctions.*

(3.4.1) *If $u \in CARA$, $S_{k+1}^u < S_k^u$.*

(3.4.2) *If $u \in CARS$, $S_{k+1}^u > S_k^u$.*

Proof. We just prove (3.4.2). The cases $k = 1, 2$ are proved in Theorem 3.1. We therefore assume $k \ge 3$. Let $u(x) = e^{\lambda x} - 1$, $\lambda > 0$. For $n \ge L$ denote by b_L the equilibrium strategy in the L-price auction $A(L, n, u, F)$, $L = k, k + 1$. Define

$S_L(v_1)$ to be the expected revenue of the seller from Buyer 1 in equilibrium, given that $\tilde{v}_1 = v_1$. By symmetry arguments it suffices to show that

(3.8) $$S_{k+1}(v_1) > S_k(v_1), \quad 0 < v_1 \leq 1.$$

Note that

For $L = k < n$, integration by parts of (3.1), where $b_L(t)F(t)^{n-L}$ serves as one of the parts, yields:

$$S_k(v_1) = \Delta(n,k) \int_{t=0}^{v_1} \left(b'_k(t)F(t) + (n-k)b_k(t)F'(t)\right) F(t)^{n-k-1} \frac{(F(v_1)-F(t))^{k-1}}{(k-1)!} dt.$$

For $L = k+1 \leq n$, (3.1) yields:

$$S_{k+1}(v_1) = \Delta(n,k)(n-k) \int_{t=0}^{v_1} b_{k+1}(t)F(t)^{n-k-1} \frac{(F(v_1)-F(t))^{k-1}}{(k-1)!} F'(t)dt.$$

Hence for $D_k(v_1) = S_{k+1}(v_1) - S_k(v_1)$,

(3.9)

$$D_k(v_1) = \Delta(n,k) \int_{t=0}^{v_1} ((n-k)(F'(t)(b_{k+1}(t) - b_k(t)) - b'_k(t)F(t))\, F(t)^{n-k-1} \frac{(F(v_1)-F(t))^{k-1}}{(k-1)!} dt.$$

We are about to prove that $D_k(v_1) > 0$ for $v_1 > 0$. Before we do it, we have to manipulate the basic equilibrium equation (3.10) similarly to what we have just done to the formulas of the seller's revenues. By Theorem A in Monderer and Tennenholtz (2000), for $L = k, k+1$,

(3.10)

$$\int_{t=0}^{x} u(x - b_L(t))F(t)^{n-L}(F(x)-F(t))^{L-3}F'(t)dt = 0, \qquad \text{for every } 0 \leq x \leq 1.$$

Integration by parts of (3.10) for $L = k$ yields:

(3.11)

$$\int_{t=0}^{v_1} (-u'(v_1 - b_k(t))b'_k(t)F(t) + (n-k)u(v_1 - b_k(t))F'(t))\, F(t)^{n-k-1} \frac{(F(v_1)-F(t))^{k-2}}{(k-2)!} dt = 0.$$

Replacing L by $k+1$ in (3.10) and multiplying both sides of the resulting equality by $\frac{(n-k)}{(k-2)!}$ yields

(3.12) $$\int_{t=0}^{v_1} ((n-k)u(v_1 - b_{k+1}(t))F'(t))\, F(t)^{n-k-1} \frac{(F(v_1)-F(t))^{k-2}}{(k-2)!} dt = 0.$$

Subtracting (3.12) from (3.11) yields:

(3.13) $$0 =$$

$$\int_{t=0}^{v_1}\!\!\left(-u'(v_1-b_k(t)){b'}_k(t)F(t)+(n-k)F'(t)(u(v_1-b_k(t))-u(v_1-b_{k+1}(t)))\right) F(t)^{n-k-1}\frac{(F(v_1)-F(t))^{k-2}}{(k-2)!}dt.$$

Since $u(x) = e^{\lambda x} - 1$ and $u'(x) = \lambda e^{\lambda x}$, (3.13) is equivalent to

(3.14)

$$\int_{t=0}^{v_1}\!\!\left(-\lambda e^{-\lambda b_k(t)}{b'}_k(t)F(t)+(n-k)F'(t)(e^{-\lambda b_k(t)}-e^{-\lambda b_{k+1}(t)})\right) F(t)^{n-k-1}\frac{(F(v_1)-F(t))^{k-2}}{(k-2)!}dt=0.$$

Differentiating (3.14) $k-1$ times with respect to v_1 yields for every $0 \leq v_1 \leq 1$

(3.15)

$$\left(-\lambda e^{-\lambda b_k(v_1)}{b'}_k(v_1)F(v_1) + (n-k)F'(v_1)(e^{-\lambda b_k(v_1)} - e^{-\lambda b_{k+1}(v_1)})\right) F(v_1)^{n-k-1} = 0.$$

By (3.15), and the facts that $\lambda > 0$ and that an equilibrium strategy is increasing, we get that $e^{-\lambda b_k(v_1)} - e^{-\lambda b_{k+1}(v_1)} > 0$ for every $v_1 > 0$. Therefore,

$$b_{k+1}(v_1) > b_k(v_1), \quad 0 < v_1 \leq 1. \tag{3.16}$$

Since the function u_λ is strictly convex,

$$e^{-\lambda b_k(v_1)} - e^{-\lambda b_{k+1}(v_1)} < \lambda e^{-\lambda b_k(v_1)}(b_{k+1}(v_1) - b_k(v_1)). \tag{3.17}$$

By applying (3.17) to (3.15) we conclude that the integrand in (3.9) is positive for $t > 0$. Hence $D_k(v_1) > 0$ for every $v_1 > 0$. □

4. Utility Equivalence. Let $A(k, n, u, G)$ be a k-price auction. For every symmetric equilibrium strategy b we denote the expected revenue of a buyer with type v in equilibrium by $B_{k,n,u,G,b}(v)$. If some of the parameters are fixed we omit them. If $A(k, n, u, F)$ is a regular auction we assume that b is the unique continuous symmetric equilibrium strategy, and we omit the subscript b.

Hon-Snir (2002) gave a general utility equivalence principle, that holds for buyers with constant absolute risk attitude. Furthermore, she showed that this equivalence principle holds if and only if the buyers have CAR. Hon-Snir's theorem does not require a continuous or symmetric setup. Using Hon-Snir's result we can prove:

Theorem 4.1. *Suppose the buyers have constant absolute attitude to risk or risk-neutrality. Assume that each of the auctions $A(k, n, u, G)$ and $A(L, n, u, G)$, $n \geq k \geq L \geq 1$, possesses an increasing symmetric equilibrium, b_k, and b_L respectively. Then*

$$B_k(v) = B_L(v), \quad v \in [0, 1]. \tag{4.1}$$

Proof. As $u \in CAR \cup RN$, we can assume without loss of generality that $u = w_\lambda$, for some real number λ. Let $s = K$ or $s = L$. By Hon-Snir (2002)

$$B_s(v) = B_s(0)e^{\lambda t} + e^{\lambda v} \int_{t=0}^{v} e^{-\lambda t} Q_s(t) dt, \tag{4.2}$$

where $Q_s(t)$ is the winning probability with respect to the equilibrium strategy b_s in $A(s, n, u, G)$. That is, for every arbitrary buyer i

$$Q_s(t) = E_G^N \left(\tau(b_s(t), (\hat{b}_s)_{-i}(\tilde{v}_{-i}) \right).$$

As b_s is increasing for $s = k, L$,

$$B_s(0) = 0, \quad \text{and} \quad Q_s(t) = E_G^N \left(\tau(t, \tilde{v}_{-i}) \right).$$

Therefore $B_k(v) = B_L(v)$. □

5. Competition in Auction Design.

Consider the following two-stage Bayesian game, which we refer to as the *auction-selection game*. There are m sellers and n buyers, $n \geq m \geq 2$. At Stage 1, each seller chooses an auction mechanism out of a class of k-price auctions, $k \in \{1, 2, ..., K\}$, $K \leq n$. At stage 2, each buyer receives his type and proceeds to choose an auction place and to submit a bid. We assume that all auctions are conducted simultaneously. In order to analyze the subgame perfect equilibria structure of this two-stage game we have to analyze the equilibria structure of its subgames. There are K^m subgames, each one is described by an m-tuple of the form $a = (a_1, \ldots, a_m)$, $a_i \in \{1, 2, ..., K\}$. That is, a denotes the subgame in which Seller s chooses a_s-price auction for every $1 \leq s \leq m$. A strategy of a buyer in each of the subgames is a vector $\sigma = (p, b) = (p_1, p_2, ..., p_m, b_1, b_2, \ldots, b_m)$, where for every location s, $p_s : [0,1] \to [0,1]$ is the probability of choosing location s at type v (hence, $\sum_{s=1}^{m} p_s(v) = 1$ for every $v \in [0,1]$), and b_s denotes the bidding strategy at location s.

In each of the subgames we restrict our attention to a buyer-symmetric equilibrium[12], σ, that satisfies some or all of the following:

E1 p_s is continuous for every location s.

E2 $b_s(0) = 0$, and b_s is increasing in $[0,1]$ for every location s.

E3 b_s is continuous for every location s.

[12]Note that we do not require equal treatment to similar auctions.

Theorem 5.1. *Assume F is a regular distribution, and the buyers have constant absolute attitude to risk or they are risk-neutral. Then, in each of the subgames $a = (a_i)_{i=1}^m$, $a_i \in \{1, 2, ..., K\}$, of the auction-selection game, at every buyer-symmetric equilibrium $\sigma = (p, b)$ that satisfies $E1 - E2$, $p_s(v) = \frac{1}{m}$ for each v and for every s.*

Before we prove this theorem we need to discuss auctions with an exogenous random participation.[13] Let $A(k, n, u, F)$ be a k-price auction with a regular distribution function. Assume that for every Buyer i, and for every type v_i, Buyer i participates in the auction with a probability $q(v_i)$, where $q : [0, 1] \to [0, 1]$ is a continuous function. Participation is not a strategic decision. $q(v_i)$ reflects the degree of awareness of buyer i to the existence of the auction. Let $\tilde{Y}^i$, $i \in N$ be $\{0, 1\}$-random variables such that the random variables $(\tilde{v}_i, \tilde{Y}^i)_{i=1}^n$ are stochastically independent, and such that

$$Prob(\tilde{Y}^i = 1 | \tilde{v}_i = v_i) = q(v_i).$$

We denote the associated Bayesian game by $A(k, n, u, F, q)$, and we refer to it as a k-price auction with random participation. As usual, b is a symmetric equilibrium strategy in $A(k, n, u, F, q)$ if for every buyer i, and for every v_i in $[0, 1]$, $b(v_i)$ maximizes the natural conditional expectation of i's utility, when each of the other buyers uses b.[14]

Let $\tilde{w}_i = \tilde{v}_i \tilde{Y}^i$. Hence, $(\tilde{w}_i)_{i \in N}$ are stochastically independent and identically distributed random variables. Let $G(x)$ be the common distribution function. That is, $G(x)$ is the probability that an arbitrary buyer i participates in the auction, and her type does not exceed x, or i does not participate in the auction —

$$G(x) = Prob(\tilde{w}_i \leq x).$$

Obviously

(5.1)

$$G(x) = 0 \quad \text{for } x < 0,\ G(1) = 1, \ \text{ and } G(x) = F(x) + \int_{t=x}^{1} (1 - q(t)) F'(t) dt, \quad 0 \leq x \leq 1.$$

Note that unless $q(v) = 1$ for all v, $G(0) > 0$. That is, the valuation function $\tilde{w}_i$ may have an atom at 0.

[13] Auctions with random number of participants are discussed in McAfee and McMillan (1987).

[14] Note that we use the common definition of Bayesian equilibrium, and thus the maximization condition should be satisfied at every v_i including types v_i for which $q(v_i) = 0$.

Lemma 5.2. *Let F be a regular distribution function. Let b, with $b(0) = 0$, be an increasing symmetric equilibrium strategy in the k-price auction with random participation $A(k, n, u, F, q)$. Then b is a symmetric strategy equilibrium in the k-price auction $A(k, n, u, G)$.*

Proof. Let i be a fixed buyer. Let $B_q(v_i, x), B(v_i, x)$ be the expected utility of i, when her type is v_i, and she bids x in $A(k, n, u, F, q)$ and $A(k, n, u, G)$ respectively. The difference between the two setups arises in the event that all other buyers do not participate, and i bids 0. In $A(k, n, u, F, q)$, i wins for sure, while in $A(k, n, u, G)$, i wins with probability $\frac{1}{n}$. Hence, for $x > 0$, $B_q(v_i, x) = B(v_i, x)$, and $B_q(v_i, 0) \geq B(v_i, 0)$. As b is in equilibrium in $A(k, n, u, F, q)$, $B_q(v_i, b(v_i)) \geq B_q(v_i, x)$ for every x. We have to show that for every v_i

$$B(v_i, b(v_i)) \geq B(v_i, x) \quad \text{for every bid } x. \tag{5.2}$$

For $v_i > 0$, $b(v_i) > b(0) = 0$. Therefore,

$$B(v_i, b(v_i)) = B_q(v_i, b(v_i)) \geq B_q(v_i, x) \geq B(v_i, x) \quad \text{ for all } x.$$

At $v_i = 0$, biding 0 is optimal in $A(k, n, u, G)$. Hence, (5.2) is satisfied. □

Proof of Theorem 5.1.

W.l.o.g. $u = w_\lambda$. Let $a = (a_i)_{i=1}^m$, $a_i \in \{1, 2, ..., K\}$ be a subgame of the auction-selection game, let $\sigma = (p, b)$ be an buyer-symmetric equilibrium in this subgame that satisfies $E1 - E2$. Therefore, for every location s, b_s is a symmetric equilibrium strategy in the a_s-price auction with random participation, $A(a_s, n, u, F, p_s)$. By Lemma 5.2, b_s is a symmetric strategy equilibrium in the a_s-price auction $A(a_s, n, u, G_s)$, where

$$G_s(x) = F(x) + \int_{t=x}^{1} (1 - p_s(t))F'(t)dt. \tag{5.3}$$

Let $0 < v_0 < 1$. We want to show that all non-zero coordinates of $p(v_0) = (p_1(v_0), \ldots p_m(v_0))$ are equal. Assume $p_a(v_0) > 0$ and $p_c(v_0) > 0$. As p_a, p_c are continuous, there exists an open interval I that contains v_0, such that $0 \notin I$ and $p_a(v), p_c(v) > 0$ for $v \in I$. Let i be a fixed buyer. As in equilibrium i assigns positive probabilities to locations a, c, her expected utility given $v \in I$ is the same at these locations. Hence, $B_a(v) = B_c(v)$ for every $v \in I$. As $B_s(0) = 0$ for all s, by (4.2)

$$B_s(v) = e^{\lambda v} \int_{t=0}^{v} e^{-\lambda t} Q_s(t)dt, \quad \text{for all } s.$$

Therefore,

$$B_s(v)e^{-\lambda v} = \int_{t=0}^{v} e^{-\lambda t} Q_s(t)dt, \quad \text{for all } s.$$

By Hon-Snir(2002), B_s is a Liptchitz function, and therefore for every s,

$$(B_s(v)e^{-\lambda v})' = Q_s(v)e^{-\lambda v} \quad \text{for Borel-almost all } v \text{ in } [0,1].$$

Therefore,

$$Q_a(v)e^{-\lambda v} = Q_b(v)e^{-\lambda v} \quad \text{for Borel-almost all } v \text{ in } I.$$

Hence,

$$Q_a(v) = Q_c(v) \quad \text{for Borel-almost all } v \text{ in } I.$$

Hence,

$$G_a(v)^{n-1} = G_c(v)^{n-1} \quad \text{for Borel-almost all } v \text{ in } I.$$

By (5.3) G_s is continuous, and therefore the above equality holds for all $v \in I$. By (5,3),

$$\int_{t=v}^{1} (1 - p_a(t))F'(t)dt = \int_{t=x}^{1} (1 - p_c(t))F'(t)dt, \quad v \in I.$$

Therefore

$$(1 - p_a(v)F'(v) = (1 - p_a(v)F'(v), \quad v \in I.$$

As F is a regular distribution function, $F'(v) > 0$ for all $v > 0$. Therefore $p_c(v) = p_a(v)$ for all $v \in I$, and in particular

$$p_a(v_0) = p_c(v_0).$$

It follows that the range of the function p is a finite set. As p is continuous, $p(v) = (p_1, ..., p_m)$ for every $v \in [0,1]$. Note that there exists $\alpha > 0$ such that $p_s = \alpha$ or $p_s = 0$ for all s. As $\sum_{s=1}^{m} p_s = 1$, there exists s such that $p_s = \alpha > 0$. Assume there exists another location, say a, such that $p_a = 0$. That is, the buyers do not go to location a. Therefore, i can deviate such that she chooses a with probability 1, and increases her expected utility to $B(v) = v$ for all v. As by (4.2) $B_s(v) < v$, $\sigma = (p, b)$ is not an equilibrium. Therefore all coordinates p_s, $1 \le s \le m$ are positive. Hence

$$p(v) = (\frac{1}{m}, ..., \frac{1}{m}) \quad \text{for all } v \in [0,1]. \quad \square$$

We are now going to discuss the sellers' strategies in the first stage of the auction selection game. A seller should choose $k \in \{1, ..., K\}$. However, independent of the other sellers' choices, she will face a k-price auction with the distribution F_m, where

$$F_m(x) = F(x) + (1 - \frac{1}{m})(1 - F(x)).$$

Hence,

$$F_m(x) = \frac{1}{m}F(x) + 1 - \frac{1}{m}.$$

Thus, the seller is just acting as a monopolist who can choose any k price auction, $1 \leq k \leq K$. F_m is not a regular distribution function. However, it is easily verified that Theorem 3.4 holds for *semi-regular* auctions, which satisfy all conditions required by regularity, except that the distribution G is required to be of the form $G(x) = \delta F(x) + (1 - \delta)$, where F is regular. Hence,

Theorem 5.3. *The following hold at every buyer-symmetric subgame perfect equilibrium σ of an auction selection game, at which all k-price auctions, $1 \leq k \leq K$, are regular, and σ satisfies $E1 - E3$ at every subgame:*

(5.3.1) *If the buyers have constant absolute risk-aversion, all sellers chooses $k = 1$.*
(5.3.2) *If the buyers have constant absolute risk-seeking, all sellers chooses $k = K$.*

Moreover, if the buyers are risk-neutral, every vector of choices for the sellers, $(a_1, ..., a_m) \in \{1, ..., K\}^m$ is sustainable in a buyer-symmetric subgame perfect equilibrium that satisfies $E1 - E2$.

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