

Pathwise Uniqueness for Stochastic Heat Equations with Hölder Continuous Coefficients: the White Noise Case

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Abstract. We prove pathwise uniqueness for solutions of parabolic stochastic pde's with multiplicative white noise if the coefficient is Hölder continuous of index $\gamma > 3/4$. The method of proof is an infinite-dimensional version of the Yamada-Watanabe argument for ordinary stochastic differential equations.

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1 Introduction

Let $\sigma : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ and consider the stochastic heat equation

$$(1.1) \quad \frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) dt + \sigma(t, x, X(t, x)) \dot{W}(x, t) + b(t, x, X(t, x)).$$

Here Δ denotes the Laplacian and \dot{W} is space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$. If $\sigma(t, x, X)$ and $b(t, x, X)$ are Lipschitz continuous in X it is well-known that there are pathwise unique solutions to (1.1) (see (Wal86)). When $\sigma(t, x, X) = \sqrt{f(t, x, X)X}$ such equations arise naturally as the scaling limits of critical branching particle systems where the branching rate at (t, x) is given by $f(t, x, X(t, x))$ and $X(t, x)$ is a measure of the local particle density at (t, x) . Such coefficients are not Lipschitz continuous and pathwise uniqueness remains open even in the case where $f \equiv 1$, $b \equiv 0$ and X is the density of super-Brownian motion (see Section III.4 of (Per02)). In this case, and more generally for $f = X^p$ for $p > 0$, uniqueness in law is known by duality arguments (see (Myt99)). The duality arguments are highly non-robust, however, and pathwise uniqueness, if true, would typically hold for a much less restrictive set of coefficients. Our goal in this work is to show pathwise uniqueness holds for solutions to (1.1) if $\sigma(t, x, \cdot)$ is Hölder continuous of order γ and $\gamma > 3/4$. The attentive reader will have already noted that the motivating example given above does not satisfy this condition.

The above equation does have the advantage of having a diagonal form—that is, when viewed as a continuum-dimensional stochastic differential equation there are no off-diagonal terms in the noise part of the equation and the diffusion coefficient for the x coordinate is a function of that coordinate alone. For finite-dimensional sde's this was the setting for Yamada and Watanabe's extension (YW71) of Itô's pathwise uniqueness results to Hölder (1/2) continuous coefficients, and so our plan will be to carry over their approach to our infinite dimensional setting. This programme was already carried out in the context of coloured noise in (MPS06), but the methods used there when specialized to white noise gives nothing beyond the classical Lipschitz uniqueness. In fact for coloured noise in higher dimensions the results in (MPS06) did not even come close to the known results on pathwise uniqueness for Lipschitz continuous coefficients (Dal99)—see the discussion after Remark 1.5 in (MPS06). This is what led to our belief that there was room for substantial improvement in the methods of (MPS06) and hence to the present work.

We introduce a growth condition, a Hölder continuity condition on σ and the standard Lipschitz condition on b :

$$(1.2) \quad \text{there exists a constant } c_{1.2} \text{ such that for all } (t, x, X) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ |\sigma(t, x, X)| + |b(t, x, X)| \leq c_{1.2}(1 + |X|),$$

$$(1.3) \quad \text{for some } \gamma > 3/4 \text{ there are } R_1, R_2 > 0 \text{ and for all } T > 0 \text{ there is an } R_0(T) \\ \text{so that for all } t \in [0, T] \text{ and all } (x, X, X') \in \mathbb{R}^3, \\ |\sigma(t, x, X) - \sigma(t, x, X')| \leq R_0(T) e^{R_1|x|} (1 + |X| + |X'|)^{R_2} |X - X'|^\gamma,$$

and

$$(1.4) \quad \text{there is a } B > 0 \text{ s.t. for all } (t, x, X, X') \in \mathbb{R}_+ \times \mathbb{R}^3, |b(t, x, X) - b(t, x, X')| \leq B|X - X'|.$$

We assume W is a white noise on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where \mathcal{F}_t satisfies the usual hypotheses. This means $W_t(\phi)$ is an \mathcal{F}_t -Brownian motion with variance $\|\phi\|_2^2$ for each $\phi \in \mathbb{L}^2(\mathbb{R}, dx)$ and $W_t(\phi)$ and $W_t(\psi)$ are independent if $\int \phi(x)\psi(x)dx = 0$. We set $p_t(x) = (2\pi t)^{-1/2} \exp\{-x^2/2t\}$, $P_t f(x) = \int f(y)p_t(y-x)dy$, and let $\mathcal{F}_t^W \subset \mathcal{F}_t$ be the filtration generated by W satisfying the usual hypotheses. A stochastic process $X : \Omega \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$, which is jointly measurable and \mathcal{F}_t -adapted, is said to be a solution to the stochastic heat equation (1.1) on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with initial condition $X_0 : \mathbb{R} \rightarrow \mathbb{R}$, if for each $t \geq 0$, and $x \in \mathbb{R}$,

$$(1.5) \quad X(t, x) = \int_{\mathbb{R}} p_t(y-x)X_0(y)dy + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)\sigma(s, y, X(s, y))W(ds, dy) \\ + \int_0^t \int_{\mathbb{R}} p_{t-s}(y-x)b(s, y, X(s, y))dyds \text{ a.s.}$$

To state the main results we introduce some notation, which will be used throughout this work: If $E \subset \mathbb{R}^d$, we write $C(E)$ for the space of continuous functions on E . A superscript k , respectively ∞ , indicates that functions are in addition k times, respectively infinitely often, continuously differentiable. A subscript b , respectively c , indicates that they are also bounded, respectively have compact support. We also define

$$\|f\|_\lambda := \sup_{x \in \mathbb{R}} |f(x)|e^{-\lambda|x|},$$

set $C_{tem} := \{f \in C(\mathbb{R}), \|f\|_\lambda < \infty \text{ for any } \lambda > 0\}$ and endow it with the topology induced by the norms $\|\cdot\|_\lambda$ for $\lambda > 0$. That is, $f_n \rightarrow f$ in C_{tem} iff $d(f, f_n) = \sum_{k=1}^{\infty} 2^{-k}(\|f - f_n\|_{1/k} \wedge 1) \rightarrow 0$ as $n \rightarrow \infty$. Then (C_{tem}, d) is a Polish space. By identifying the white noise W , with the associated Brownian sheet, we may view W as a stochastic processes with sample paths in $C(\mathbb{R}_+, C_{tem})$. Here as usual, $C(\mathbb{R}_+, C_{tem})$ is given the topology of uniform convergence on compacts.

A stochastically weak solution to (1.1) is a solution on some filtered space with respect to some noise W , i.e., the noise and space are not specified in advance.

With this notation we can state the following standard existence result whose proof is a minor modification of Theorem 1.2 of (MPS06) and is given in the next Section.

Theorem 1.1 *Let $X_0 \in C_{tem}$, and let $b, \sigma : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfy (1.2), (1.3), and (1.4). Then there exists a stochastically weak solution to (1.1) with sample paths a.s. in $C(\mathbb{R}_+, C_{tem})$.*

We say pathwise uniqueness holds for solutions of (1.1) in $C(\mathbb{R}_+, C_{tem})$ if for every deterministic initial condition, $X_0 \in C_{tem}$, any two solutions to (1.1) with sample paths a.s. in $C(\mathbb{R}_+, C_{tem})$ must be equal with probability 1. For Lipschitz continuous σ , this follows from Theorem 2.2 of (Shi94). Here then is our main result:

Theorem 1.2 *Assume that $b, \sigma : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfy (1.2), (1.3) and (1.4). Then pathwise uniqueness holds for solutions of (1.1) in $C(\mathbb{R}_+, C_{tem})$.*

As an immediate consequence of Theorems 1.1 and 1.2 we get existence and uniqueness of strong solutions and joint uniqueness in law of (X, W) .

Theorem 1.3 *Assume that $b, \sigma : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfy (1.2), (1.3) and (1.4). Then for any $X_0 \in C_{tem}$ there is a solution X to (1.1) on $(\Omega, \mathcal{F}_\infty^W, \mathcal{F}_t^W, \mathbb{P})$ with sample paths a.s. in $C(\mathbb{R}_+, C_{tem})$. If X' is any other solution to (1.1) on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with sample paths a.s. in $C(\mathbb{R}_+, C_{tem})$, then $X(t, x) = X'(t, x)$ for all t, x a.s. The joint law P_{X_0} of (X, W) on $C(\mathbb{R}_+, C_{tem})^2$ is uniquely determined by X_0 and is Borel measurable in X_0 .*

Proof. The Borel measurability of the law is proved as in Exercise 6.7.4 in (SV79). We now apply Theorem 3.14 of (Kur07), with the Polish state spaces S_1 and S_2 for the driving process (W) and solution (X) in that work both equal to $C(\mathbb{R}_+, C_{tem})$. Theorems 1.1 and 1.2 imply the hypotheses of weak existence and pointwise uniqueness of (a) of that result. The conclusions of an \mathcal{F}_t^W -adapted (strong) solution and uniqueness in law of (X, W) follow from the conclusions in Theorem 3.14 (b) (of (Kur07)) of a strong compatible solution and joint uniqueness in law, respectively. (Note that Lemma 3.11 of the above reference shows that a strong, compatible solution must be \mathcal{F}_t^W -adapted.) Finally the equality of X and X' is immediate from Theorem 1.2 on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ as X is also a solution for the larger filtration. \blacksquare

Remark 1.4 (a) When assuming (1.2), it suffices to assume (1.3) for $|X - X'| \leq 1$. Indeed this condition is immediate from (1.2) for $|X - X'| \geq 1$ with $R_1 = 0$ and $R_2 = 1$.

(b) (1.3) implies the local Hölder condition:

$$(1.6) \quad \text{for some } \gamma > 3/4 \text{ for all } K > 0 \text{ there is an } L_K \text{ so that for all } t \in [0, K] \\ \text{and } x, X_1, X_2 \in [-K, K], |\sigma(t, x, X_1) - \sigma(t, x, X_2)| \leq L_K |X_1 - X_2|^\gamma.$$

In fact it prescribes the growth rate of the Hölder constants L_K (polynomial in X and exponential in x).

In order to give a bit of intuition for Theorem 1.2, we recall the result from (MPS06) which dealt with the stochastic heat equation driven by *coloured* noise. Let $\dot{W}(t, x)$ be the mean zero Gaussian noise on $\mathbb{R}_+ \times \mathbb{R}^d$ with covariance given by

$$(1.7) \quad E \left[\dot{W}(t, x) \dot{W}(s, y) \right] = \delta_0(t - s) k(x - y),$$

where

$$(1.8) \quad k(x - y) \leq c|x - y|^{-\alpha},$$

for some $\alpha \in (0, d \wedge 2)$. Note that the white noise considered in this paper is the case $k(x) = \delta_0(x)$. It formally corresponds to $\alpha = 1$ in dimension $d = 1$. Now let X satisfies the SPDE:

$$(1.9) \quad \frac{\partial}{\partial t} X(t, x) = \frac{1}{2} \Delta X(t, x) dt + \sigma(X(t, x)) \dot{W}(x, t),$$

with \dot{W} being the coloured noise just described. Then the following result was proved in (MPS06).

Theorem 1.5 ((MPS06)) For $\alpha < 2\gamma - 1$, pathwise uniqueness holds for (1.9).

Let $\tilde{u} = X^1 - X^2$ be the difference of two solutions to (1.9). The proof of Theorem 1.5 relied on a study of the Hölder continuity of $\tilde{u}(t, \cdot)$ at points where $\tilde{u}(t, x)$ is “small”. Let ξ be the Hölder exponent of $\tilde{u}(t, \cdot)$ at such points. The following connection between parameter ξ and the pathwise uniqueness was shown in (MPS06) (see condition (41) in the proof of Theorem 4.1 there): If

$$(1.10) \quad \alpha < \xi(2\gamma - 1),$$

then pathwise uniqueness holds for (1.9). Hence, the better the regularity one has for \tilde{u} near its zero set, the “weaker” the hypotheses required for pathwise uniqueness. It was shown in (MPS06) that at the points x where $\tilde{u}(t, x)$ is “small”, $\tilde{u}(t, \cdot)$ is Hölder continuous with any exponent ξ such that

$$(1.11) \quad \xi < \frac{1 - \frac{\alpha}{2}}{1 - \gamma} \wedge 1.$$

(For the precise statement of this result see Theorem 2.2 in the next section.) Note that in the case when $\alpha < 2\gamma - 1$, (1.11) turns into the following condition

$$(1.12) \quad \xi < 1,$$

and this together with (1.10) imply Theorem 1.5.

Now assume \dot{W} is white noise on $\mathbb{R}_+ \times \mathbb{R}$ and $d = 1$. This formally corresponds to the $\alpha = 1$, and in this case the conditions (1.10), (1.11) can be written as

$$(1.13) \quad 1 < \xi(2\gamma - 1),$$

$$(1.14) \quad \xi < \frac{1}{2(1 - \gamma)} \wedge 1.$$

For $\gamma \geq 1/2$, we have $\frac{1}{2(1 - \gamma)} \geq 1$ and hence one can take $\xi < 1$ arbitrarily close to 1 (a proof of this is given in Theorem 2.3 below), substitute it into (1.13), and get a vacuous condition for pathwise uniqueness, namely

$$\gamma > 1.$$

To improve on this we will need to get more refined information on the difference, u , of two solutions to (1.1) near the points x_0 where $u(t, x_0) \approx 0$. To be more precise, suppose one is able to show that

$$(1.15) \quad |u(t, x)| \leq c|x - x_0|^\xi$$

for any

$$(1.16) \quad \xi < \frac{1}{2(1 - \gamma)} \wedge 2.$$

By substituting the upper bound for ξ from (1.16) into (1.13) and doing a bit of arithmetic one gets the following condition for pathwise uniqueness

$$(1.17) \quad \gamma > 3/4,$$

which is the result claimed in Theorem 1.2. We will in fact verify a version of (1.15) under (1.16) and $\gamma > 3/4$. A more detailed description of our approach, is given in Section 2.

The above discussion allows us to conjecture a stronger result on pathwise uniqueness for the case of equations driven by a coloured noise:

Conjecture 1.6 *If*

$$(1.18) \quad \alpha < 2(2\gamma - 1),$$

then pathwise uniqueness holds for (1.9).

The reasoning for this conjecture is similar to that for the white noise case. Let \tilde{u} again be the difference of two solutions to (1.9). Suppose that if $\tilde{u}(t, x_0) \approx 0$ then at the points nearby we have

$$(1.19) \quad \tilde{u}(t, x) \leq c|x - x_0|^\xi$$

for any

$$(1.20) \quad \xi < \frac{1 - \frac{\alpha}{2}}{1 - \gamma} \wedge 2.$$

By substituting the upper bound for ξ from (1.20) into (1.10) and simple algebra one gets (1.18) as a condition for pathwise uniqueness for (1.9). Note that (1.18) can be equivalently written as

$$(1.21) \quad \gamma > \frac{1}{2} + \frac{\alpha}{4}.$$

In the next Section we give a quick proof of Theorem 1.1 and then turn to the main result, Theorem 1.2. Following the natural analogue of the Yamada-Watanabe argument for stochastic pde's, as in (MPS06), the problem quickly reduces to one of showing that the analogue of the local time term is zero (Proposition 2.1). As described above, the key ingredient here will be tight control on the spatial behaviour of the difference of two solutions, when this difference is very small, that is, when the solutions separate. Roughly speaking, as in Yamada and Watanabe's argument we first show that solutions must separate in a gentlemanly manner and therefore cannot separate at all. Section 2 includes a heuristic description of the method and further explanation of why $\gamma = 3/4$ is critical in our approach. It also gives an outline of the contents of the entire paper.

Convention on Constants. Constants whose value is unimportant and may change from line to line are denoted c_1, c_2, \dots , while constants whose values will be referred to later and appear initially in say, Lemma i.j are denoted $c_{i,j}$ or $C_{i,j}$. Dependence of constants on R_i or B will be suppressed in our notation.

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2 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1 This is standard so we only give a sketch and set $b \equiv 0$ for simplicity. By taking weak limits as $T \rightarrow \infty$ we may assume $R_0(T) = R_0$ is independent of T . Choose a symmetric $\psi_n \in C_c^\infty$ so that $0 \leq \psi_n \leq 1$, $\|\psi_n'\|_\infty \leq 1$, $\psi_n(x) = 1$ if $|x| \leq n$ and $\psi_n(x) = 0$ if $|x| \geq n + 2$. Let

$$\sigma_n(t, x, X) = \int \sigma(t, x, X') p_{2^{-n}}(X' - X) dX' \psi_n(X).$$

It is easy to then check the following:

$$(2.22) \quad |\sigma_n(t, x, X)| \leq 2c_{1.2}(1 + |X|),$$

$$(2.23) \quad |\sigma_n(t, x, X') - \sigma_n(t, x, X)| \leq c_n|X' - X|,$$

and

$$(2.24) \quad |\sigma_n(t, x, X) - \sigma(t, x, X)| \leq c_{2.24} \left[e^{R_1|x|} (|X|^{R_2} + 1) 2^{-n\gamma/2} + (1 + |X|)(1 - \psi_n(|X|)) \right] \\ \rightarrow 0 \quad \text{uniformly on compacts as } n \rightarrow \infty.$$

Use (2.22), (2.23) and Theorem 2.2 of (Shi94) to see there are solutions X^n to (1.5)_n (all with respect to W)—here (1.5)_n is (1.5) but with σ_n in place of σ . Now argue as in Section 6 of (Shi94) (see the proof of Theorem 2.2) or in the derivation of Theorem 1.2 of (MPS06) (the present white noise setting simplifies those arguments) to see that $\{X^n\}$ is tight in $C(\mathbb{R}_+, C_{tem})$. More specifically, using the growth condition (1.2), it is straightforward to carry over the proof of Proposition 1.8(a) of (MPS06) (see Lemmas A.3 and A.5 of that paper) and show

$$(2.25) \quad \text{for all } T, \lambda, p > 0, \quad \sup_n E \left(\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |X^n(t, x)|^p e^{-\lambda|x|} \right) < \infty.$$

The above bounds in turn give uniform bounds on the p th moments of the space-time increments of X^n (see Lemma A.4 of (MPS06)) and hence tightness. Indeed, the orthogonality of white noise makes all these calculation somewhat easier. By Skorohod's theorem we may assume X^{n_k} converges a.s. to X in $C(\mathbb{R}_+, C_{tem})$ on some probability space. It is now easy to use (2.24) to see that (perhaps on a larger space), X solves (1.5). \blacksquare

Next consider Theorem 1.2 and assume its hypotheses throughout. By Remark 1.4(a) decreasing γ only weakens the hypotheses and so we may, and shall, assume that

$$(2.26) \quad 3/4 < \gamma < 1.$$

Let X^1 and X^2 be two solutions of (1.5) on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with sample paths in $C(\mathbb{R}_+, C_{tem})$ a.s., with the same initial condition, $X^1(0) = X^2(0) = X_0 \in C_{tem}$, and of course the same noise W . For adapted processes with sample paths in $C(\mathbb{R}_+, C_{tem})$, (1.5) is equivalent to the distributional form of (1.1) (see Theorem 2.1 of (Shi94)). That is, for $i = 1, 2$ and $\Phi \in C_c^\infty(\mathbb{R})$:

$$(2.27) \quad \int_{\mathbb{R}} X^i(t, x) \Phi(x) dx = \int_{\mathbb{R}} X_0^i(x) \Phi(x) dx + \int_0^t \int_{\mathbb{R}} X^i(s, x) \frac{1}{2} \Delta \Phi(x) dx ds \\ + \int_0^t \int_{\mathbb{R}} \sigma(s, x, X^i(s, x)) \Phi(x) W(ds, dx) \\ + \int_0^t \int_{\mathbb{R}} b(s, x, X^i(s, x)) \Phi(x) dx ds \quad \forall t \geq 0 \quad a.s.$$

Let

$$(2.28) \quad T_K = \inf \{ s \geq 0 : \sup_y (|X^1(s, y)| \vee |X^2(s, y)|) e^{-|y|} > K \} \wedge K.$$

We first show that the hypothesis (1.3) may be strengthened to

$$(2.29) \quad \begin{aligned} &\text{for some } 1 > \gamma > 3/4 \text{ there are } R_0, R_1 \geq 1 \text{ so that for all } t \geq 0 \\ &\text{and all } (x, X, X') \in \mathbb{R}^3, |\sigma(t, x, X) - \sigma(t, x, X')| \leq R_0 e^{R_1|x|} |X - X'|^\gamma. \end{aligned}$$

Assume that Theorem 1.2 holds under (2.29) and that σ satisfies (1.3). Define

$$\sigma_K(t, x, X) = \sigma(t, x, (X \vee (-Ke^{|x|})) \wedge Ke^{|x|}) 1(t \leq K).$$

Then

$$|\sigma_K(t, x, X) - \sigma_K(t, x, X')| \leq R_0(K) e^{R_1|x|} (1 + 2Ke^{|x|})^{R_2} |X - X'|^\gamma,$$

and so (2.29) holds with $R_1 + R_2$ in place of R_1 (the restriction that $R_i \geq 1$ is for convenience and is no restriction). Providing that for $\lambda = 1$, $\|X_0\|_\lambda < K$, we have

$$(2.30) \quad \sigma(t, x, X^i(t, x)) = \sigma_K(t, x, X^i(t, x)) \text{ for all } x \text{ and } t \leq T_K.$$

Therefore σ_K satisfies (2.29) and of course (1.2). So we may apply Theorem 1.3 to the equation (2.27)_K with σ_K in place of σ . Let P_{K, X_0} be the unique law of the solution (X, W) on $C(\mathbb{R}_+, C_{tem})^2$. Using the Borel measurability of these laws in X_0 , it is now easy to “piece together” solutions to construct solutions \tilde{X}^i , $i = 1, 2$ to (2.27)_K starting at X_0 on some probability space such that $(\tilde{X}^1(\cdot \wedge T_K), \tilde{X}^2(\cdot \wedge T_K))$ is equal in law to $(X^1(\cdot \wedge T_K), X^2(\cdot \wedge T_K))$. By pathwise uniqueness in (2.27)_K we get $\tilde{X}^1 = \tilde{X}^2$ and so $X^1(\cdot \wedge T_K) = X^2(\cdot \wedge T_K)$. Letting $K \rightarrow \infty$ gives $X^1 = X^2$, as required.

We now follow the approach in Section 2 of (MPS06) and reduce the theorem to showing the analogue of the “local time term” in the Yamada-Watanabe proof is zero. Let

$$a_n = \exp\{-n(n+1)/2\}$$

so that

$$(2.31) \quad a_{n+1} = a_n e^{-n-1} = a_n a_n^{2/n}.$$

Define functions $\psi_n \in C_c^\infty(\mathbb{R})$ such that $\text{supp}(\psi_n) \subset (a_n, a_{n-1})$, and

$$(2.32) \quad 0 \leq \psi_n(x) \leq \frac{2}{nx} \quad \text{for all } x \in \mathbb{R} \text{ as well as } \int_{a_n}^{a_{n-1}} \psi_n(x) dx = 1.$$

Finally, set

$$(2.33) \quad \phi_n(x) = \int_0^{|x|} \int_0^y \psi_n(z) dz dy.$$

From this it is easy to see that $\phi_n(x) \uparrow |x|$ uniformly in x . Note that each ψ_n , and thus also each ϕ_n , is identically zero in a neighborhood of zero. This implies that $\phi_n \in C^\infty(\mathbb{R})$ despite the absolute value in its definition. We have

$$(2.34) \quad \phi_n'(x) = \text{sgn}(x) \int_0^{|x|} \psi_n(y) dy,$$

$$(2.35) \quad \phi_n''(x) = \psi_n(|x|).$$

Thus, $|\phi'_n(x)| \leq 1$, and $\int \phi''_n(x)h(x)dx \rightarrow h(0)$ for any function h which is continuous at zero.

Define

$$u \equiv X^1 - X^2.$$

Let $\Phi \in C_c^\infty(\mathbb{R})$ satisfy $0 \leq \Phi \leq 1$, $\text{supp}(\Phi) \subset (-1, 1)$ and $\int_{\mathbb{R}} \Phi(x)dx = 1$, and set $\Phi_x^m(y) = m\Phi(m(x - y))$. Let $\langle \cdot, \cdot \rangle$ denote the scalar product on $\mathbb{L}^2(\mathbb{R})$. By applying Itô's Formula to the semimartingales $\langle X_t^i, \Phi_x^m \rangle$ in (2.27) it follows that

$$\begin{aligned} & \phi_n(\langle u_t, \Phi_x^m \rangle) \\ &= \int_0^t \int_{\mathbb{R}} \phi'_n(\langle u_s, \Phi_x^m \rangle) (\sigma(s, y, X^1(s, y)) - \sigma(s, y, X^2(s, y))) \Phi_x^m(y) W(ds, dy) \\ &+ \int_0^t \phi'_n(\langle u_s, \Phi_x^m \rangle) \langle u_s, \frac{1}{2} \Delta \Phi_x^m \rangle ds \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{R}} \psi_n(|\langle u_s, \Phi_x^m \rangle|) (\sigma(s, y, X^1(s, y)) - \sigma(s, y, X^2(s, y)))^2 \\ &\quad \times \Phi_x^m(y)^2 dy ds \\ &+ \int_0^t \int_{\mathbb{R}} \phi'_n(\langle u_s, \Phi_x^m \rangle) (b(s, y, X^1(s, y)) - b(s, y, X^2(s, y))) \Phi_x^m(y) dy ds. \end{aligned}$$

We integrate this function of x against another non-negative test function $\Psi \in C_c^\infty([0, t_0] \times \mathbb{R})$ ($t_0 \in (0, \infty)$). Choose $K_1 \in \mathbb{N}$ large so that for $\lambda = 1$,

$$(2.36) \quad \|X_0\|_\lambda < K_1 \text{ and } \Gamma \equiv \{x : \Psi_s(x) > 0 \text{ for some } s \leq t_0\} \subset (-K_1, K_1).$$

We then obtain by the classical and stochastic versions of Fubini's Theorem (see Theorem 2.6 of (Wal86) for the latter), and arguing as in the proof of Proposition II.5.7 of (Per02) to handle the time dependence in Ψ , that for any $t \in [0, t_0]$,

$$\begin{aligned} (2.37) \quad & \langle \phi_n(\langle u_t, \Phi_x^m \rangle), \Psi_t \rangle \\ &= \int_0^t \int_{\mathbb{R}} \langle \phi'_n(\langle u_s, \Phi_x^m \rangle) \Phi_x^m(y), \Psi_s \rangle (\sigma(s, y, X^1(s, y)) - \sigma(s, y, X^2(s, y))) W(ds, dy) \\ &+ \int_0^t \langle \phi'_n(\langle u_s, \Phi_x^m \rangle) \langle u_s, \frac{1}{2} \Delta \Phi_x^m \rangle, \Psi_s \rangle ds \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^2} \psi_n(|\langle u_s, \Phi_x^m \rangle|) (\sigma(s, y, X^1(s, y)) - \sigma(s, y, X^2(s, y)))^2 \\ &\quad \times \Phi_x^m(y)^2 dy \Psi_s(x) dx ds + \int_0^t \langle \phi_n(\langle u_s, \Phi_x^m \rangle), \dot{\Psi}_s \rangle ds \\ &+ \int_0^t \int_{\mathbb{R}} \langle \phi'_n(\langle u_s, \Phi_x^m \rangle) \Phi_x^m(y), \Psi_s \rangle (b(s, y, X^1(s, y)) - b(s, y, X^2(s, y))) dy ds \\ &\equiv I_1^{m,n}(t) + I_2^{m,n}(t) + I_3^{m,n}(t) + I_4^{m,n}(t) + I_5^{m,n}(t). \end{aligned}$$

(2.38)

The expectation condition in Walsh's Theorem 2.6 may be realized by localization, using the stopping times $\{T_K\}$.

Set $m_n = a_{n-1}^{-1/2} = \exp\{(n-1)n/4\}$ for $n \in \mathbb{N}$. In the integral defining $I_3^{m_{n+1}, n+1}$ we may assume $|x| \leq K_1$ by (2.36) and so $|y| \leq K_1 + 1$. Let $K \in \mathbb{N}^{\geq K_1} = \{K_1, K_1 + 1, \dots\}$. If $s \leq T_K$, then for such a y ,

$$|X^i(s, y)| \leq Ke^{|y|} \leq Ke^{(K_1+1)} \text{ for } i = 1, 2.$$

Therefore (1.6), (2.32) and (2.31) show that if $K' = Ke^{(K_1+1)} (\geq K_1 + 1)$, then for all $t \in [0, t_0]$,

$$\begin{aligned} & I_3^{m_{n+1}, n+1}(t \wedge T_K) \\ & \leq \frac{1}{2} \int_0^{t \wedge T_K} \int \int 2(n+1)^{-1} |\langle u_s, \Phi_x^{m_{n+1}} \rangle|^{-1} 1(a_{n+1} < |\langle u_s, \Phi_x^{m_{n+1}} \rangle| < a_n) \\ & \quad \times L_{K'}^2 |u(s, y)|^{2\gamma} m_{n+1} \Phi_x^{m_{n+1}}(y) \Psi_s(x) dy dx ds \\ & \leq L_{K'}^2 a_{n+1}^{-1} a_n^{-1/2} \int_0^{t \wedge T_K} \int \int 1(a_{n+1} < |\langle u_s, \Phi_x^{m_{n+1}} \rangle| < a_n) |u(s, y)|^{2\gamma} \Phi_x^{m_{n+1}}(y) \Psi_s(x) dy dx ds \\ (2.39) & = L_{K'}^2 a_n^{-3/2-2/n} \int_0^{t \wedge T_K} \int \int 1(a_{n+1} < |\langle u_s, \Phi_x^{m_{n+1}} \rangle| < a_n) |u(s, y)|^{2\gamma} \Phi_x^{m_{n+1}}(y) \Psi_s(x) dy dx ds. \end{aligned}$$

We define

$$(2.40) \quad I^n(t) = a_n^{-3/2-2/n} \int_0^t \int \int 1(|\langle u_s, \Phi_x^{m_{n+1}} \rangle| < a_n) |u(s, y)|^{2\gamma} \Phi_x^{m_{n+1}}(y) \Psi_s(x) dy dx ds.$$

Proposition 2.1 *Suppose $\{U_{M,n,K} : M, n, K \in \mathbb{N}, K \geq K_1\}$ are \mathcal{F}_t -stopping times such that for each $K \in \mathbb{N}^{\geq K_1}$,*

$$(2.41) \quad (H_1) \quad U_{M,n,K} \leq T_K, \quad U_{M,n,K} \uparrow T_K \text{ as } M \rightarrow \infty \text{ for each } n, \text{ and} \\ \lim_{M \rightarrow \infty} \sup_n P(U_{M,n,K} < T_K) = 0,$$

and

$$(2.42) \quad (H_2) \quad \text{For all } M \in \mathbb{N} \text{ and } t_0 > 0, \lim_{n \rightarrow \infty} E(I^n(t_0 \wedge U_{M,n,K})) = 0.$$

Then the conclusion of Theorem 1.2 holds.

Proof. We adapt the reasoning in Lemma 2.2 of (MPS06) for the coloured noise setting to our white noise driven equation. As in (2.25) we have

$$(2.43) \quad \text{for all } T, \lambda, p > 0, \quad E\left(\sup_{0 \leq t \leq T} \sup_{x \in \mathbb{R}} |u(t, x)|^p e^{-\lambda|x|}\right) < \infty.$$

Let

$$Z_n(t) = \int \phi_n(\langle u_t, \Phi_x^{m_n} \rangle) \Psi_t(x) dx.$$

Fix $K \in \mathbb{N}^{\geq K_1}$ and $0 \leq t \leq t_0$. Note that since $0 \leq \phi_n(z) \leq |z|$ and $\Psi \geq 0$,

$$\begin{aligned}
(2.44) \quad 0 \leq Z_n(t \wedge T_K) &\leq \int \int |u(t \wedge T_K, y)| \Phi_x^{m_n}(y) \Psi(t \wedge T_K, x) dy dx \\
&\leq 2K \int \int e^{|y|} \Phi_x^{m_n}(y) \Psi(t \wedge T_K, x) dy 1(|x| \leq K_1) dx \\
&\leq 2K e^{K_1+1} c_1(\Psi).
\end{aligned}$$

With (2.43) in hand, the proof of Lemma 2.2(a) of (MPS06) is easily adapted (again it is in fact easier) to show

$$(2.45) \quad \{I_1^{m_n, n}(s) : s \leq t_0\} \text{ is an } \mathbb{L}^2\text{-bounded sequence of } \mathbb{L}^2 \text{ martingales.}$$

The proof of Lemma 2.2(b) of (MPS06) applies directly to show that

$$I_2^{m_n, n} = I_2^{m_n, n, 1} + I_2^{m_n, n, 2},$$

where for any stopping time T ,

$$(2.46) \quad I_2^{m_n, n, 1}(t \wedge T) \rightarrow \int_0^{t \wedge T} \int |u(s, x)| \frac{1}{2} \Delta \Psi_s(x) dx ds \text{ in } \mathbb{L}^1 \text{ as } n \rightarrow \infty$$

(again the key bound here is (2.43)), and we have the one-sided bound

$$(2.47) \quad I_2^{m_n, n, 2}(s) \leq \frac{a_n}{n} C(\Psi) \text{ for all } s \leq t_0 \text{ and } n.$$

(In the notation of (MPS06), $I_2^{m_n, n, 1}(s) = \int_0^s I_{2,3}^{m_n, n}(r) dr$ and $I_2^{m_n, n, 2}(s) = \int_0^s I_{2,1}^{m_n, n}(r) + I_{2,2}^{m_n, n}(r) dr$.) The proof of Lemma 2.2(c) of (MPS06) also applies directly to show that for any stopping time T ,

$$(2.48) \quad I_4^{m_n, n}(t \wedge T) \rightarrow \int_0^{t \wedge T} \int |u(s, x)| \dot{\Psi}_s(x) dx ds \text{ in } \mathbb{L}^1 \text{ as } n \rightarrow \infty.$$

Since $|\phi'_n| \leq 1$, (1.4) implies that for a stopping time T ,

$$(2.49) \quad I_5^{m_n, n}(t \wedge T) \leq B \int_0^{t \wedge T} \int \int |u(s, y)| \Phi_x^{m_n}(y) \Psi_s(x) dy dx ds \equiv B \tilde{I}_5^n(t \wedge T).$$

It follows easily from (2.43) that $\{\tilde{I}_5^n(t_0) : n \in \mathbb{N}\}$ is \mathbb{L}^2 -bounded and, as $n \rightarrow \infty$,

$$(2.50) \quad \tilde{I}_5^n(t \wedge T) \rightarrow \int_0^{t \wedge T} \int |u(s, x)| \Psi_s(x) dx ds \text{ a.s. and hence in } \mathbb{L}^1 \text{ by the above.}$$

Let $\varepsilon > 0$. Then (H_1) , (2.44), (2.45), (2.46) and (2.48) show that, by a standard result for uniformly integrable random variables, there is an M_0 so that

$$\begin{aligned}
(2.51) \quad \sup_n E \left(\left(|Z_n(t \wedge T_K)| + |I_1^{m_n, n}(t \wedge T_K)| + |I_2^{m_n, n, 1}(t \wedge T_K)| + |I_4^{m_n, n}(t \wedge T_K)| \right) \right. \\
\left. \times 1(U_{M, n, K} < T_K) \right) < \varepsilon \text{ for all } M \geq M_0.
\end{aligned}$$

From (2.37), (2.49), the non-negativity of $I_3^{m_n, n}$ and \tilde{I}_5^n , and Fatou's Lemma, we have for $M \geq M_0$,

$$\begin{aligned}
& E\left(\int |u(t \wedge T_K, x)| \Psi_{t \wedge T_K}(x) dx\right) \\
& \leq \liminf_{n \rightarrow \infty} E(Z_n(t \wedge T_K) 1(U_{M, n, K} = T_K)) + E(Z_n(t \wedge T_K) 1(U_{M, n, K} < T_K)) \\
& \leq \liminf_{n \rightarrow \infty} E(I_1^{m_n, n}(t \wedge T_K)) + E(I_2^{m_n, n, 1}(t \wedge T_K)) + E(I_2^{m_n, n, 2}(t \wedge T_K) 1(U_{M, n, K} = T_K)) \\
& \quad + E(I_3^{m_n, n}(t \wedge U_{M, n, K})) + E(I_4^{m_n, n}(t \wedge T_K)) + BE(\tilde{I}_5^n(t \wedge T_K)) \\
& \quad - E(I_1^{m_n, n}(t \wedge T_K) 1(U_{M, n, K} < T_K)) - E(I_2^{m_n, n, 1}(t \wedge T_K) 1(U_{M, n, K} < T_K)) \\
& \quad - E(I_4^{m_n, n}(t \wedge T_K) 1(U_{M, n, K} < T_K)) + E(Z_n(t \wedge T_K) 1(U_{M, n, K} < T_K)) \\
& \leq E\left(\int_0^{t \wedge T_K} \int |u(s, x)| \frac{1}{2} \Delta \Psi_s(x) dx ds\right) \\
& \quad + E\left(\int_0^{t \wedge T_K} \int |u(s, x)| \dot{\Psi}_s(x) dx ds\right) + BE\left(\int_0^{t \wedge T_K} \int |u(s, x)| \Psi_s(x) dx ds\right) + \varepsilon,
\end{aligned}$$

by (2.45), (2.46), (2.47), (H_2) (together with the bound (2.39)), (2.48) (2.50), and (2.51), respectively. Let $\varepsilon \downarrow 0$ to see that

$$E\left(\int |u(t \wedge T_K, x)| \Psi_{t \wedge T_K}(x) dx\right) \leq E\left(\int_0^{t \wedge T_K} \int |u(s, x)| \left(\frac{1}{2} \Delta \Psi_s(x) + \dot{\Psi}_s(x) + B \Psi_s(x)\right) dx ds\right).$$

Let $K \rightarrow \infty$ and use Dominated Convergence (recall (2.43)) on each side to conclude that

$$\int E(|u(t, x)|) \Psi_t(x) dx \leq \int_0^t \int E(|u(s, x)|) \left(\frac{1}{2} \Delta \Psi_s(x) + \dot{\Psi}_s(x) + B \Psi_s(x)\right) dx ds, \quad 0 \leq t \leq t_0.$$

This gives (34) of (MPS06) with an additional drift term $B \Psi_s(x)$. One proceeds exactly as in Section 3 of that reference, using the semigroup $e^{Bt} P_t$ in place of P_t , to see that since $E(|u(t, x)|)$ is a finite (by (2.43)) non-negative subsolution of the heat equation with initial data zero, therefore $E(|u(t, x)|) = 0$ and so $X^1 = X^2$ by continuity of paths. \blacksquare

The construction of $\{U_{M, n, K}\}$ and verification of (H_1) and (H_2) will be the objective of the rest of this work.

Notation. For $t, t' \geq 0$ and $x, x' \in \mathbb{R}$ let $d((t, x), (t', x')) = \sqrt{|t' - t|} + |x' - x|$.

Note that the indicator function in the definition of I^n implies there is an $\hat{x}_0 \in (x - \sqrt{a_n}, x + \sqrt{a_n})$ such that $|u(s, \hat{x}_0)| \leq a_n$. If we could take $\hat{x}_0 = y$ we could bound $I_n(t)$ by $C(t) a_n^{-3/2 - 2/n + 2\gamma}$, and (H_1) and (H_2) would follow immediately with $U_{M, n, K} = T_K$. (The criticality of $3/4$ in this argument is illusory as it follows from our choice of m_n .) The hypotheses of Proposition 2.1 now turn on getting good bounds on $|u(s, y) - u(s, \hat{x}_0)|$. The standard $1/2 - \varepsilon$ -Hölder modulus¹ not surprisingly,

¹Although this is well-known ‘folklore’ result we were not able to find the exact reference. One can easily check that the estimates in the proof of Corollary 3.4 in (Wal86) give $1/2 - \varepsilon$ -Hölder spatial modulus; similarly the result of (SS02) can be immediately extended to cover the white noise case; in both works the Lipschitz assumptions on noise coefficients can be relaxed to linear growth assumptions and the proofs still go through.

gives nothing. In (MPS06) this was refined to a $1 - \varepsilon$ -Hölder modulus *near points where u is small* as we now describe. Let

$$Z(N, K)(\omega) = \{(t, x) \in [0, T_K] \times [-K, K] : \text{there is a } (\hat{t}_0, \hat{x}_0) \in [0, T_K] \times \mathbb{R} \text{ such that} \\ d((\hat{t}_0, \hat{x}_0), (t, x)) \leq 2^{-N}, \text{ and } |u(\hat{t}_0, \hat{x}_0)| \leq 2^{-N}\}.$$

Return now to the SPDE driven by coloured noise (1.9) from Section 1. Let \tilde{u} be the difference of two solutions of (1.9) and $\alpha \in (0, 2 \wedge d)$ be the covariance kernel exponent as in (1.8). Let $\tilde{Z}(N, K)$ be defined as $Z(N, K)$ with \tilde{u} instead of u . The following improved modulus of continuity was proved in (MPS06) (see Theorem 4.1 and the first two paragraphs of the proof of Corollary 4.2 in that reference).

Theorem 2.2 *For each $K \in \mathbb{N}$ and $0 < \xi < \frac{1-\alpha}{1-\gamma} \wedge 1$ there is an $N_0 = N_0(\xi, K, \omega) \in \mathbb{N}$ a.s. such that for all natural numbers $N \geq N_0$ and all $(t, x) \in \tilde{Z}(N, K)$,*

$$d((t', x'), (t, x)) \leq 2^{-N} \text{ and } t' \leq T_K \text{ implies } |\tilde{u}(t', x') - \tilde{u}(t, x)| \leq 2^{-N\xi}.$$

In the white noise setting the result holds with $\alpha = 1$. Recall u is the difference of two solutions to (1.1).

Theorem 2.3 *Assume $\gamma \geq 1/2$. For each $K \in \mathbb{N}$ and $\xi \in (0, 1)$ there is an $N_0 = N_0(\xi, K, \omega) \in \mathbb{N}$ a.s. such that for all natural numbers $N \geq N_0$ and all $(t, x) \in Z(N, K)$,*

$$d((t', x'), (t, x)) \leq 2^{-N} \text{ and } t' \leq T_K \text{ implies } |u(t', x') - u(t, x)| \leq 2^{-N\xi}.$$

Proof. The proof of Theorem 2.2 applies with $\alpha = 1$ (the dependance of σ on t, x alters nothing in the proof). In fact it is now considerably simpler because of the orthogonality of white noise increments. The required tools are Lemma 4.1 and Lemma 4.3 below. With this choice of α and $\gamma \geq 1/2$, the upper bound on ξ in Theorem 2.2 becomes 1. In (MPS06) there was no drift term, but the calculations for the Lipschitz drift term are simpler still. Here one uses Hölder's inequality to utilize the \mathbb{L}^2 bounds in Lemma 4.3. ■

The proof of Theorem 1.2 is long and involved so before descending into the technical details of the derivation of (H_1) and (H_2) , we now give a heuristic description of the method with $b \equiv 0$ throughout, and also try to explain why $\gamma = 3/4$ is critical in our approach. The choice of $m_n = a_{n-1}^{-1/2}$ appears arbitrarily in the above so let us for the moment set $m_n = a_{n-1}^{-\alpha_0} \approx a_n^{-\alpha_0}$ for some $\alpha_0 > 0$. (H_1) and (H_2) are delicate ways of ensuring $I^n(t)$ approaches zero as $n \rightarrow \infty$ and so our goal is to show that

$$(2.52) \quad I^n(t) \approx a_n^{-1-\alpha_0} \int_0^t \int \int 1(|\langle u_s, \Phi_x^{m_{n+1}} \rangle| < a_n) |u(s, y)|^{2\gamma} \Phi_x^{m_{n+1}}(y) \Psi_s(x) dy dx ds \\ \rightarrow 0 \text{ as } n \rightarrow \infty.$$

We have taken (and will take) some small liberties with the “local time term” $I^n(t)$ (with this new choice of m_n) in the first line. In the integrand in (2.52) the variable y must be within $2a_n^{\alpha_0}$ of a

point \hat{x}_0 where $|u(s, \hat{x}_0)| < a_n$. If we simply replace y with \hat{x}_0 , $I^n(t)$ is at most

$$a_n^{-1-\alpha_0} \int_0^t \int \int a_n^{2\gamma} \Phi_x^{m_{n+1}}(y) \Psi_s(x) dy dx ds \leq C t a_n^{2\gamma-1-\alpha_0} \rightarrow 0,$$

if $\gamma > 1/2$ and α_0 is small enough. This is a bit too crude but shows it will be crucial to get good estimates on $u(s, \cdot)$ near points where it is small (and also shows we are already forced to assume $\gamma > 1/2$). Theorem 2.3 implies that

$$(2.53) \quad \gamma \geq 1/2 \text{ implies } u(t, \cdot) \text{ is } \xi\text{-H\"older continuous near its zero set for } \xi < 1,$$

and so allows us to bound $|u(s, y) - u(s, \hat{x}_0)|$. Use this in (2.52) and take $0 < \alpha_0 \leq 1$ to bound $I^n(t)$ by

$$\begin{aligned} a_n^{-1-\alpha_0} \int_0^t \int \int C a_n^{\alpha_0 \xi^{2\gamma}} \Phi_x^{m_{n+1}}(y) \Psi_s(x) dy dx ds \\ \leq C t a_n^{-1+\alpha_0(\xi^{2\gamma-1})} \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

if $\gamma > 1$ and we choose α_0, ξ close to one. Of course $\gamma > 1$ is not a viable choice but this shows we are now getting close, and in fact in the coloured noise setting of (MPS06) the above argument sufficed for the results there, although there was some work to be done to implement this idea carefully.

To increase our control on $u(s, \cdot)$ near its zero set we will improve (2.53) to

$$(2.54) \quad \gamma > 3/4 \text{ implies } u'(s, \cdot) \text{ is } \xi\text{-H\"older on } \{x : u(s, x) \approx u'(s, x) \approx 0\} \text{ for } \xi < 1,$$

where u' denotes the spatial derivative. Corollary 5.9 below with $m = \bar{m} + 1$ is the closest result which comes to a formal statement of the above, although the condition on γ is implicit. We hasten to add that the literal existence of these derivatives is not established, nor expected in general. We will shortly give a rigorous interpretation to these statements but let us continue heuristically first.

We first make the case that for $\gamma < 3/4$, we cannot expect the following slight strengthening of (2.54):

$$(2.55) \quad u(s, \cdot) \text{ is } C^2 \text{ on } \{x : u(s, x) \approx u'(s, x) \approx 0\}.$$

A formal differentiation of (1.5) (recall $b \equiv 0$ and u is the difference of the X^i 's) gives for $u(t, x) \approx u'(t, x) \approx 0$,

$$(2.56) \quad u''(t, x) = \int_0^t \int p_{t-s}''(y-x) [\sigma(s, y, X^1(s, y)) - \sigma(s, y, X^2(s, y))] W(ds, dy).$$

If σ is a Weierstrass-type function that realizes its Hölder modulus at typical points we have

$$|\sigma(s, y, X^1(s, y)) - \sigma(s, y, X^2(s, y))| \approx L |u(s, y)|^\gamma,$$

and for $s < t$ and very close to t , we have by a Taylor series expansion in space,

$$|u(s, y)| \approx |u''(s, x)| \frac{(y-x)^2}{2}.$$

Use these approximations in the finite square function associated with the right hand side of (2.56) and conclude that

$$\begin{aligned}
\infty &> \int_{t-\delta}^t \int p''_{t-s}(y-x)^2 \left[\frac{|u''(s,x)|(y-x)^2}{2} \right]^{2\gamma} dy ds \\
&\approx c|u''(t,x)|^{2\gamma} \int_{t-\delta}^t \int p_{t-s}(z)^2 [z^2(t-s)^{-2} - (t-s)^{-1}]^2 z^{4\gamma} dz ds \\
&\approx c|u''(t,x)|^{2\gamma} \int_{t-\delta}^t (t-s)^{2\gamma-\frac{5}{2}} ds,
\end{aligned}$$

which implies $\gamma > 3/4$. (The last line uses $p_u(z)^2 \leq u^{-1/2}p_u(z)$ and scaling.)

We next show how (2.54) will lead to (2.52). Taking further liberties with $I^n(t)$ and recalling $m_n \approx a_n^{-\alpha_0}$, we get

$$\begin{aligned}
(2.57) \quad I^n(t) &\approx a_n^{-1-\alpha_0} \sum_{\beta} \int_0^t \int \int 1(|u(s,x) \leq a_n, u'(s,x) \approx \pm a_n^{\beta}) |u(s,y)|^{2\gamma} \Phi_x^{m_{n+1}}(y) \Psi_s(x) dy dx ds \\
&\equiv \sum_{\beta} I_{\beta}^n(t),
\end{aligned}$$

where \sum_{β} indicates we are summing over a finite grid $\beta_i \in [0, \bar{\beta}]$ ($\bar{\beta}$ to be determined below) and are partitioning space-time according to the which interval $\pm[a_n^{\beta_i+1}, a_n^{\beta_i})$ contains the value of $u'(s,x)$ (write $u'(s,x) \approx \pm a_n^{\beta_i}$)—the end values of β_i being interpreted in the natural manner. As the sum is finite we may fix $\beta \in (0, \bar{\beta}]$ and consider only $u'(s,x) \approx a_n^{\beta}$; the negative values of β may be handled in a symmetric manner. The value $\beta = 0$ is a bit special but should be clear from the argument below. A Taylor series expansion and (2.54) with $\xi \approx 1$ show that for y as in the integrand of $I_{\beta}^n(t)$,

$$\begin{aligned}
|u(s,y)| &\leq |u(s,x)| + (|u'(s,x)| + M|y-x|^{\xi})|y-x| \\
&\leq a_n + a_n^{\beta+\alpha_0} + M a_n^{\alpha_0(\xi+1)} \\
&\leq C a_n^{(\frac{\xi}{2} \wedge \beta) + \frac{1}{2}},
\end{aligned}$$

where a comparison of the first and last terms in the second line leads naturally to $\alpha_0 = 1/2$. Substitute this into the integrand of I_{β}^n , integrate out y , and conclude

$$(2.58) \quad I_{\beta}^n(t) \leq C a_n^{\frac{-3}{2} + \gamma + \gamma(2\beta \wedge \xi)} \int_0^t \int 1(|u(s,x)| \leq a_n, u'(s,x) \approx a_n^{\beta}) \Psi_s(x) dx ds.$$

For $\beta = \bar{\beta}$ the precise meaning of $u'(s,x) \approx a_n^{\bar{\beta}}$ is $0 \leq u'(s,x) \leq a_n^{\bar{\beta}}$ and we have from (2.58),

$$(2.59) \quad I_{\bar{\beta}}^n(t) \leq C t a_n^{\frac{-3}{2} + \gamma + \gamma(2\bar{\beta} \wedge \xi)}.$$

Consider $0 < \beta < \bar{\beta}$. Recall that $\{x : \Psi_s(x) > 0 \text{ for some } s \leq t_0\} \subset [-K_1, K_1]$, let

$$S_n(s) = \{x \in [-K_1, K_1] : |u(s,x)| \leq a_n, u'(s,x) \geq a_n^{\beta}\}$$

and $|S_n(s)|$ denote the Lebesgue measure of $S_n(s)$. From (2.54) we see that if $x \in S_n(s)$, then $u'(s, y) \geq \frac{a_n^\beta}{2}$ if $|y - x| \leq M^{-1}a_n^{\beta/\xi}$, and so by the Fundamental Theorem of Calculus,

$$u(s, y) > a_n \text{ if } 4a_n^{1-\beta} < |y - x| \leq L^{-1}a_n^{\beta/\xi}.$$

A simple covering argument now shows that $|S_n(s)| \leq c(M, K_1)a_n^{1-\beta}a_n^{-\beta/\xi}$ and (2.58) implies

$$(2.60) \quad \begin{aligned} I_\beta^n(t) &\leq Cta_n^{\frac{-3}{2} + \gamma + \gamma(2\bar{\beta} \wedge \xi) + 1 - \beta - \frac{\beta}{\xi}} \\ &\leq Cta_n^{\gamma(1 + (2\bar{\beta} \wedge \xi)) - \frac{1}{2} - \bar{\beta}(1 + \frac{1}{\xi})}. \end{aligned}$$

So from (2.59) and (2.60) we see that $\lim_{n \rightarrow \infty} I_\beta^n(t) = 0$ will follow for all $\beta \leq \bar{\beta}$ if

$$\gamma(1 + (2\bar{\beta} \wedge 1)) > \frac{3}{2} \quad \text{and} \quad \gamma(1 + (2\bar{\beta} \wedge 1)) > \frac{1}{2} + 2\bar{\beta},$$

that is, $\gamma > (1 + (2\bar{\beta} \wedge 1))^{-1}(\frac{3}{2} \vee (\frac{1}{2} + 2\bar{\beta}))$. The right-hand side is minimized when $\bar{\beta} = \frac{1}{2}$, and leads to $\gamma > \frac{3}{4}$, as required, and also establishes the range $0 \leq \beta \leq \frac{1}{2}$, which will be used below.

The above heuristics show that $\gamma > 3/4$ and the regularity of u given in (2.54) (or (2.55)) is optimal for our approach. If we try weakening the regularity condition on u , the above discussion shows we would have to increase $3/4$ to show that $I^n(t) \rightarrow 0$. The earlier discussion shows that a strengthening of the regularity on u would require increasing $3/4$ as well.

A major obstruction to (2.54) is the fact that we cannot expect $u'(s, x)$ to exist as soon as $u(s, x) \neq 0$ (and don't even know this is the case for $u(s, x) = 0$). So instead, if $D(r, y) = \sigma(r, y, X^1(r, y)) - \sigma(r, y, X^2(r, y))$, then we will use (1.5) to decompose u as

$$(2.61) \quad \begin{aligned} u(t, x) &= \int_0^{t-a_n} \int p_{t-r}(y-x)D(r, y)W(dr, dy) + \int_{t-a_n}^t \int p_{t-r}(y-x)D(r, y)W(dr, dy) \\ &\equiv u_{1, a_n}(t, x) + u_{2, a_n}(t, x). \end{aligned}$$

u_{1, a_n} is smooth in the spatial variable and so the above arguments may be applied with $u'_{1, a_n}(t, x)$ playing the role of $u'(t, x)$, while u_{2, a_n} and its increments should lead to small and manageable error terms. Proposition 5.14 gives the required bounds on the increments of u_{2, a_n^α} , and (as noted above) Corollary 5.9 is the analogue of (2.54) for u'_{1, a_n^α} ($\alpha \in [0, 1]$). (The reason for the extension to a_n^α is discussed below.) The proofs of these results are incorporated into an inductive proof of a space-time bound (P_m) for $u(t, x)$ when (t, x) is close to a point (\hat{t}_0, \hat{x}_0) where

$$(2.62) \quad |u(\hat{t}_0, \hat{x}_0)| \leq a_n \text{ and } |u'_{1, a_n^\alpha}(\hat{t}_0, \hat{x}_0)| \leq a_n^\beta.$$

If

$$(2.63) \quad d = \sqrt{|t - \hat{t}_0|} + |x - \hat{x}_0|,$$

then, roughly speaking, (P_m) bounds $|u(t, x)|$ by

$$(2.64) \quad d^\xi [d^{\tilde{\gamma}_m - 1} + a_n^\beta],$$

where $\tilde{\gamma}_m$ increases in m and equals 2 for m large, and $\xi < 1$ as usual. When $\tilde{\gamma}_m = 2$ this does capture the kind of bound one expects from (2.54). The reader can find a precise statement of (P_m) in (5.5) prior to Proposition 5.1 (the statement of its validity).

The $m = 0$ case will be an easy consequence of our improved local modulus of continuity, Theorem 2.3. Note that (2.29) implies

$$(2.65) \quad |D(r, y)| \leq R_0 e^{R_1|y|} |u(r, y)|^\gamma.$$

The inductive proof of (P_m) proceeds by using (2.65) and then (2.64) to bound the square functions associated with the space-time increments of u'_{1,a_n^α} and u_{2,a_n^α} for points near (\hat{t}_0, \hat{x}_0) as in (2.62) (recall (2.61)). These give good control of the integrands of these square functions near the points where they have singularities. This will then lead to Corollary 5.9 and Proposition 5.14, our “ m th order” bounds for the increments of u'_{1,a_n^α} and u_{2,a_n^α} . We then use the slightly generalized version of (2.61),

$$(2.66) \quad u(t, x) = u_{1,a_n^\alpha}(t, x) + u_{2,a_n^\alpha}(t, x)$$

to derive (P_{m+1}) . At this point we will optimize over α since decreasing α increases the regularity of u_{1,a_n^α} but increases the size of the error term u_{2,a_n^α} . The optimal choice will be so that $a_n^\alpha \approx d$, where d is as in (2.63).

There are at least two issues to address here. First, how do you control $u'_{1,a_n^\alpha}(t, x)$ when all you know is $|u'_{1,a_n^\alpha}(t, x)| \leq a_n^\beta$? Second, how do you control the *time* increments of u_{1,a_n^α} when you only have good estimates on the spatial derivatives? The first question is answered in Proposition 5.11 which will give surprisingly good bounds on $|u'_{1,a_n^\alpha}(\hat{t}_0, \hat{x}_0) - u'_{1,a_n^\alpha}(\hat{t}_0, \hat{x}_0)|$. The second question is answered in Proposition 5.13, where the key step is to note (see (5.73)) that for $t > t'$,

$$|u_{1,a_n^\alpha}(t, x) - u_{1,a_n^\alpha}(t', x)| \approx |P_{t-t'}(u_{1,a_n^\alpha}(t', \cdot))(x) - u_{1,a_n^\alpha}(t', x)|,$$

where P_t is the Brownian semigroup. The fact that the Brownian semigroup, $P_t f$, inherits temporal regularity from spatial regularity of f will give the required regularity in time.

A critical step in the above argument was finding a form of (P_m) which actually iterates to produce (P_{m+1}) . Note also that although the required bound on $I_\beta^n(t)$ (see (2.58)) only required good spatial estimates for $u(s, \cdot)$ near points $(s, x) = (\hat{t}_0, \hat{x}_0)$ as in (2.62), the iteration of estimates requires an expansion in both space and time.

Turning now to a brief description of the contents of the paper, we first set $b \equiv 0$. In Section 3 we reduce (H_1) and (H_2) (that is (2.41) and (2.42)) to a result (Proposition 3.3) on control of the spatial increments of u_{2,a_n^α} and size of u'_{1,a_n^α} on relatively long intervals near a spatial point where $|u(s, \hat{x}_0)|$ is small and $u'_{1,a_n^\alpha}(s, \hat{x}_0) \approx a_n^\beta$. This includes the covering argument sketched above. Section 4 gives some integral bounds for heat kernels and their derivatives which will help bound the square functions of the increments of u'_{1,a_n^α} and u_{2,a_n^α} . The heart of the proof of Theorem 1.2 is given in Section 5 where the inductive proof of (P_m) is given. As was sketched above, this argument includes good local expansions for u'_{1,a_n^α} and u_{2,a_n^α} near points where $|u|$ and $|u'_{1,a_n^\alpha}|$ are small, although the (easier) proof for u_{2,a_n^α} is deferred until Section 7. These expansions, with m large enough, are then used in Section 6 to prove Proposition 3.3 and so complete the proof of Theorem 1.2 for $b \equiv 0$. In Section 8 we describe the relatively simple additions that are needed to include a Lipschitz drift b in the argument already presented.

3 Verification of the Hypotheses of Proposition 2.1

We assume throughout this Section that $b \equiv 0$ —the relatively simple refinements required to include the drift are outlined in Section 8. Let X^1, X^2 be as in Section 2, $u = X^1 - X^2$, and assume the hypotheses of Theorem 1.2 as well as (2.29). If $D(s, y) = \sigma(s, y, X^1(s, y)) - \sigma(s, y, X^2(s, y))$, then

$$(3.1) \quad u(t, x) = \int_0^t \int p_{t-s}(y-x) D(s, y) W(ds, dy) \text{ a.s. for all } (t, x),$$

and by (2.29),

$$(3.2) \quad |D(s, y)| \leq R_0 e^{R_1 |y|} |u(s, y)|^\gamma.$$

δ will always take values in $(0, 1]$. Recall that (P_t) is the Brownian semigroup, and let

$$(3.3) \quad u_{1,\delta}(t, x) = P_\delta(u_{(t-\delta)^+})(x) \text{ and } u_{2,\delta}(t, x) = u(t, x) - u_{1,\delta}(t, x).$$

Since $P_\delta : C_{tem} \rightarrow C_{tem}$ is uniformly continuous (by Lemma 6.2(ii) of (Shi94)), $u_{1,\delta}$ and $u_{2,\delta}$ both have sample paths in $C(\mathbb{R}, C_{tem})$. (3.1) implies that

$$u_{1,\delta}(t, x) = \int \left[\int_0^{(t-\delta)^+} \int p_{(t-\delta)^+-s}(y-z) D(s, y) W(ds, dy) \right] p_\delta(z-x) dz.$$

A stochastic Fubini argument (Theorem 2.6 of (Wal86)) then gives

$$(3.4) \quad u_{1,\delta}(t, x) = \int_0^{(t-\delta)^+} \int p_{t-s}(y-x) D(s, y) W(ds, dy) \text{ a.s. for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

(The above identity is trivial for $t \leq \delta$ since $u(0, \cdot) \equiv 0$.) The expectation condition in Walsh's Theorem 2.6 may be realized by localization with the stopping times $\{T_K\}$, working with $D(s \wedge T_K)$, and letting $K \rightarrow \infty$. It follows that

$$(3.5) \quad u_{2,\delta}(t, x) = \int_{(t-\delta)^+}^t \int p_{t-s}(y-x) D(s, y) W(ds, dy) \text{ a.s. for all } (t, x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Hence $u_{j,\delta}$ ($j = 1, 2$) define jointly continuous versions of the right-hand sides of (3.4) and (3.5).

Notation. If $s, t \geq 0$ and $x \in \mathbb{R}$, let $G_\delta(s, t, x) = P_{(t-s)^+ + \delta}(u_{(s-\delta)^+})(x)$ and $F_\delta(s, t, x) = -\frac{d}{dx} G_\delta(s, t, x) \equiv -G'_\delta(s, t, x)$, if the derivative exists.

Lemma 3.1 $G'_\delta(s, t, x)$ exists for all $(s, t, x) \in \mathbb{R}_+^2 \times \mathbb{R}$, is jointly continuous in (s, t, x) , and satisfies

$$(3.6) \quad F_\delta(s, t, x) = \int_0^{(s-\delta)^+} \int p'_{(t \vee s)-r}(y-x) D(r, y) W(dr, dy) \text{ for all } s \text{ a.s. for all } (t, x).$$

Proof. Since $G_\delta(s, t, x) = \int p_{(t-s)^++\delta}(y-x)u((s-\delta)^+, y)dy$ and

$$(3.7) \quad \sup_{s \leq T, y} e^{-|y|} |u((s-\delta)^+, y)| < \infty \text{ for all } T > 0 \text{ a.s.},$$

a simple Dominated Convergence argument shows that

$$(3.8) \quad G'_\delta(s, t, x) = - \int p'_{(t-s)^++\delta}(y-x)u((s-\delta)^+, y)dy \quad \text{for all } (s, t, x) \text{ a.s.}$$

Another application of (3.7) and Dominated Convergence gives the a.s. joint continuity of the right-hand side of (3.8), and hence of G'_δ .

To prove (3.6) we may assume without loss of generality that $t \geq s > \delta$. From (3.8) and (3.1) we have w.p. 1,

$$G'_\delta(s, t, x) = - \int p'_{t-s+\delta}(y-x) \left[\int_0^{s-\delta} \int p_{s-\delta-r}(z-y)D(r, z)W(dr, dz) \right] dy \text{ a.s.}$$

Now use the stochastic Fubini theorem, as in the derivation of (3.4) above, to see that

$$\begin{aligned} G'_\delta(s, t, x) &= - \int_0^{s-\delta} \int \left[\int p'_{t-s+\delta}(y-x)p_{s-\delta-r}(z-y)dy \right] D(r, z)W(dr, dz) \\ &= - \int_0^{s-\delta} \int p'_{t-r}(z-x)D(r, z)W(dr, dz) \quad \text{a.s.} \end{aligned}$$

In the last line we have used Dominated Convergence yet again to differentiate through the integral in the Chapman-Kolmogorov equation. As both sides of (3.6) are continuous in s we may take the null set to be independent of s . ■

Remark 3.2 Since $G_\delta(t, t, x) = u_{1,\delta}(t, x)$, as a special case of the above we see that $u'_{1,\delta}(t, x)$ is a.s. jointly continuous and satisfies

$$(3.9) \quad u'_{1,\delta}(t, x) = - \int_0^{(t-\delta)^+} \int p'_{t-s}(y-x)D(s, y)W(ds, dy) \quad \text{a.s. for all } (t, x).$$

Definition. For $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\begin{aligned} \hat{x}_n(t, x)(\omega) &= \inf \{ y \in [x - \sqrt{a_n}, x + \sqrt{a_n}] : |u(t, y)| = \inf \{ |u(t, z)| : |z - x| \leq \sqrt{a_n} \} \\ &\quad \in [x - \sqrt{a_n}, x + \sqrt{a_n}]. \end{aligned}$$

It is easy to use the continuity of u to check that \hat{x}_n is well-defined and $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \times \mathcal{F}$ -measurable.

We fix a $K_0 \in \mathbb{N}^{\geq K_1}$ and positive constants satisfying

$$(3.10) \quad 0 < \varepsilon_1 < \frac{1}{100}(\gamma - \frac{3}{4}), \quad 0 < \varepsilon_0 < \frac{\varepsilon_1}{100}.$$

We introduce a grid of β values by setting

$$L = L(\varepsilon_0, \varepsilon_1) = \lfloor ((1/2) - 6\varepsilon_1)/\varepsilon_0 \rfloor,$$

and

$$(3.11) \quad \begin{aligned} \beta_i &= i\varepsilon_0 \in [0, \frac{1}{2} - 6\varepsilon_1], \quad \alpha_i = 2(\beta_i + \varepsilon_1) \in [0, 1], \quad i = 0, \dots, L, \\ \beta_{L+1} &= \frac{1}{2} - \varepsilon_1. \end{aligned}$$

Note that $\beta = \beta_i, i = 0, \dots, L + 1$, satisfies

$$(3.12) \quad 0 \leq \beta \leq \frac{1}{2} - \varepsilon_1$$

If $s \geq 0$ set

$$J_{n,0}(s) = \left\{ x : |x| \leq K_0, |\langle u_s, \Phi_x^{m_{n+1}} \rangle| \leq a_n, u'_{1,a_n}(s, \hat{x}_n(s, x)) \geq \frac{a_n^{\varepsilon_0}}{4} \right\},$$

$$J_{n,L}(s) = \left\{ x : |x| \leq K_0, |\langle u_s, \Phi_x^{m_{n+1}} \rangle| \leq a_n, u'_{1,a_n}(s, \hat{x}_n(s, x)) \in [0, \frac{a_n^{\beta_L}}{4}] \right\},$$

and for $i = 1, \dots, L - 1$ set

$$J_{n,i}(s) = \left\{ x : |x| \leq K_0, |\langle u_s, \Phi_x^{m_{n+1}} \rangle| \leq a_n, u'_{1,a_n}(s, \hat{x}_n(s, x)) \in [\frac{a_n^{\beta_{i+1}}}{4}, \frac{a_n^{\beta_i}}{4}] \right\}.$$

If $t_0 > 0$ is as in (H_2) (in (2.42)) and $i = 0, \dots, L$, define

$$J_{n,i} = \left\{ (s, x) : 0 \leq s, x \in J_{n,i}(s) \right\},$$

and if $0 \leq t \leq t_0$, let

$$I_i^n(t) = a_n^{-\frac{3}{2} - \frac{2}{n}} \int_0^t \int \int 1_{J_{n,i}(s)}(x) |u(s, y)|^{2\gamma} \Phi_x^{m_{n+1}}(y) \Psi_s(x) dy dx ds.$$

Let

$$\begin{aligned} I_+^n(t) &= a_n^{-\frac{3}{2} - \frac{2}{n}} \int_0^t \int \int 1(u'_{1,a_n}(s, \hat{x}_n(s, x)) \geq 0) 1(|\langle u_s, \Phi_x^{m_{n+1}} \rangle| < a_n) \\ &\quad \times |u(s, y)|^{2\gamma} \Phi_x^{m_{n+1}}(y) \Psi_s(x) dy dx ds. \end{aligned}$$

Then to prove Proposition 2.1 it suffices to construct the stopping times $\{U_{M,n} \equiv U_{M,n,K_0} : M, n \in \mathbb{N}\}$ satisfying (H_1) (in (2.41)) such that

$$(3.13) \quad \text{for each } M \in \mathbb{N}, \lim_{n \rightarrow \infty} E(I_+^n(t_0 \wedge U_{M,n})) = 0.$$

Note that (3.13) implies (H_2) (in (2.42)) by symmetry (interchange X_1 and X_2).

Our definitions imply

$$I_+^n(t) \leq \sum_{i=0}^L I_i^n(t) \text{ for all } t \leq t_0,$$

and so to prove (H_2) it suffices to show that for $i = 0, \dots, L$,

$$(H_{2,i}) \quad \text{for all } M \in \mathbb{N}, \quad \lim_{n \rightarrow \infty} E(I_i^n(t_0 \wedge U_{M,n})) = 0.$$

Notation. $\bar{l}_n(\beta) = a_n^{\beta+5\varepsilon_1}$.

Now introduce the related sets:

$$\begin{aligned} \tilde{J}_{n,0}(s) = & \left\{ x \in [-K_0, K_0] : |\langle u_s, \Phi_x^{m_{n+1}} \rangle| \leq a_n, u'_{1,a_n^{\alpha_0}}(s, x') \geq a_n^{\beta_1}/16 \right. \\ & \left. \text{for all } x' \in [x - 5\bar{l}_n(\beta_0), x + 5\bar{l}_n(\beta_0)], \right. \\ & |u_{2,a_n^{\alpha_0}}(s, x') - u_{2,a_n^{\alpha_0}}(s, x'')| \leq 2^{-75} a_n^{\beta_1} (|x' - x''| \vee a_n^{\gamma-2\beta_0(1-\gamma)-\varepsilon_1}) \\ & \left. \text{for all } x' \in [x - 4\sqrt{a_n}, x + 4\sqrt{a_n}], x'' \in [x' - \bar{l}_n(\beta_0), x' + \bar{l}_n(\beta_0)], \right. \\ & \left. \text{and } |u(s, x')| \leq 3a_n^{(1-\varepsilon_0)/2} \text{ for all } x' \in [x - \sqrt{a_n}, x + \sqrt{a_n}] \right\}, \\ \tilde{J}_{n,L}(s) = & \left\{ x \in [-K_0, K_0] : |\langle u_s, \Phi_x^{m_{n+1}} \rangle| \leq a_n, |u'_{1,a_n^{\alpha_L}}(s, x')| \leq a_n^{\beta_L} \right. \\ & \left. \text{for all } x' \in [x - 5\bar{l}_n(\beta_L), x + 5\bar{l}_n(\beta_L)], \right. \\ & \text{and } |u_{2,a_n^{\alpha_L}}(s, x') - u_{2,a_n^{\alpha_L}}(s, x'')| \leq 2^{-75} a_n^{\beta_L+1} (|x' - x''| \vee a_n^{\gamma-2\beta_L(1-\gamma)-\varepsilon_1}) \\ & \left. \text{for all } x' \in [x - 4\sqrt{a_n}, x + 4\sqrt{a_n}], x'' \in [x' - \bar{l}_n(\beta_L), x' + \bar{l}_n(\beta_L)] \right\}, \end{aligned}$$

and for $i \in \{1, \dots, L-1\}$,

$$\begin{aligned} \tilde{J}_{n,i}(s) = & \left\{ x \in [-K_0, K_0] : |\langle u_s, \Phi_x^{m_{n+1}} \rangle| \leq a_n, u'_{1,a_n^{\alpha_i}}(s, x') \in [a_n^{\beta_{i+1}}/16, a_n^{\beta_i}] \right. \\ & \left. \text{for all } x' \in [x - 5\bar{l}_n(\beta_i), x + 5\bar{l}_n(\beta_i)], \right. \\ & \text{and } |u_{2,a_n^{\alpha_i}}(s, x') - u_{2,a_n^{\alpha_i}}(s, x'')| \leq 2^{-75} a_n^{\beta_{i+1}} (|x' - x''| \vee a_n^{\gamma-2\beta_i(1-\gamma)-\varepsilon_1}) \\ & \left. \text{for all } x' \in [x - 4\sqrt{a_n}, x + 4\sqrt{a_n}], x'' \in [x' - \bar{l}_n(\beta_i), x' + \bar{l}_n(\beta_i)] \right\}, \end{aligned}$$

Finally for $0 \leq i \leq L$, set

$$\tilde{J}_{n,i} = \{(s, x) : s \geq 0, x \in \tilde{J}_{n,i}(s)\}.$$

Notation. $n_M(\varepsilon_1) = \inf\{n \in \mathbb{N} : a_n^{\varepsilon_1} \leq 2^{-M-8}\}$, $n_0(\varepsilon_0, \varepsilon_1) = \sup\{n \in \mathbb{N} : \sqrt{a_n} < 2^{-a_n^{-\varepsilon_0\varepsilon_1/4}}\}$, where $\sup \emptyset = 1$.

The following proposition will be proved in Section 6.

Proposition 3.3 $\tilde{J}_{n,i}(s)$ is a compact set for all $s \geq 0$. There exist stopping times $\{U_{M,n} \equiv U_{M,n,K_0} : M, n \in \mathbb{N}\}$ satisfying (H_1) from Proposition 2.1 such that for $i \in \{0, \dots, L\}$, $\tilde{J}_{n,i}(s)$ contains $J_{n,i}(s)$ for all $0 \leq s < U_{M,n}$ and

$$(3.14) \quad n > n_M(\varepsilon_1) \vee n_0(\varepsilon_0, \varepsilon_1).$$

Since $K_0 \in \mathbb{N}^{\geq K_1}$ was arbitrary this proposition implies that there exist stopping times satisfying (H_1) such that the inclusion $\tilde{J}_{n,i}(s) \supset J_{n,i}(s)$ holds up to these stopping times for n sufficiently

large. This inclusion means that given a value of the derivative of u_{1,a_n} at *some* point in a *small* neighborhood $[x - \sqrt{a_n}, x + \sqrt{a_n}]$ of x where $|u|$ is small, one can guarantee that the derivative of u_{1,a_n} (for a certain α) is of the same order at *any* point in a much *larger* neighborhood of x (note that $\bar{l}_n(\beta) \gg \sqrt{a_n}$ for $\beta \leq \frac{1}{2} - 6\varepsilon_1$). Moreover we can also control u_{2,a_n^α} on those long intervals. Our goal now is to show that this implies $(H_{2,i})$ for $i = 0, \dots, L$. The next three lemmas provide necessary tools for this.

Throughout the rest of the section we may, and shall, assume that the parameters $M, n \in \mathbb{N}$ satisfy (3.14), although the importance of n_0 in (3.14) will not be clear until Section 6.

Lemma 3.4 *Assume $i \in \{0, \dots, L\}$, $x \in \tilde{J}_{n,i}(s)$ and $|x' - x| \leq 4\sqrt{a_n}$.*

- (a) *If $i > 0$, then $|u(s, x'') - u(s, x')| \leq 2a_n^{\beta_i} (|x'' - x'| \vee a_n^{\gamma-2\beta_i(1-\gamma)-\varepsilon_1})$ for all $|x'' - x'| \leq \bar{l}_n(\beta_i)$.*
(b) *If $i < L$, and $a_n^{\gamma-2\beta_i(1-\gamma)-\varepsilon_1} \leq |x'' - x'| \leq \bar{l}_n(\beta_i)$, then*

$$u(s, x'') - u(s, x') \begin{cases} \geq 2^{-5} a_n^{\beta_i+1} (x'' - x') & \text{if } x'' \geq x', \\ \leq 2^{-5} a_n^{\beta_i+1} (x'' - x') & \text{if } x'' \leq x'. \end{cases}$$

Proof. (a) For n, i, s, x, x', x'' as in (a), we have (since $\beta_i + 5\varepsilon_1 < \frac{1}{2}$)

$$(3.15) \quad |x' - x| \vee |x'' - x| \leq 5\bar{l}_n(\beta_i).$$

We can therefore apply the definition of $\tilde{J}_{n,i}$ and the Mean Value Theorem to conclude that

$$\begin{aligned} |u(s, x'') - u(s, x')| &\leq |u_{1,a_n^{\alpha_i}}(s, x'') - u_{1,a_n^{\alpha_i}}(s, x')| + |u_{2,a_n^{\alpha_i}}(s, x'') - u_{2,a_n^{\alpha_i}}(s, x')| \\ &\leq a_n^{\beta_i} |x'' - x'| + 2^{-75} a_n^{\beta_i+1} (|x'' - x'| \vee a_n^{\gamma-2\beta_i(1-\gamma)-\varepsilon_1}) \\ &\leq 2a_n^{\beta_i} (|x'' - x'| \vee a_n^{\gamma-2\beta_i(1-\gamma)-\varepsilon_1}). \end{aligned}$$

(b) Consider n, i, s, x, x', x'' as in (b) with $x'' \geq x'$. We again have (3.15) and can argue as in (a) to see that

$$\begin{aligned} u(s, x'') - u(s, x') &= u_{1,a_n^{\alpha_i}}(s, x'') - u_{1,a_n^{\alpha_i}}(s, x') + u_{2,a_n^{\alpha_i}}(s, x'') - u_{2,a_n^{\alpha_i}}(s, x') \\ &\geq (a_n^{\beta_i+1}/16)(x'' - x') - 2^{-75} a_n^{\beta_i+1} (x'' - x') \\ &\geq (a_n^{\beta_i+1}/32)(x'' - x'). \end{aligned}$$

The case $x'' \leq x'$ is similar. ■

Notation. $l_n(\beta_i) = (65a_n^{1-\beta_i+1}) \vee a_n^{\gamma-2\beta_i(1-\gamma)-\varepsilon_1}$, $F_n(s, x) = \int \Phi^{m_{n+1}}(y - x)u(s, y)dy$.

Lemma 3.5 *Assume $i \in \{0, \dots, L-1\}$ and $(s, x) \in \tilde{J}_{n,i}$.*

- (a) *If $l_n(\beta_i) \leq |x - \tilde{x}| \leq \bar{l}_n(\beta_i)$, then*

$$F_n(s, \tilde{x}) - F_n(s, x) \begin{cases} \geq 2^{-5} a_n^{\beta_i+1} (\tilde{x} - x) & \text{if } \tilde{x} \geq x, \\ \leq 2^{-5} a_n^{\beta_i+1} (\tilde{x} - x) & \text{if } \tilde{x} \leq x. \end{cases}$$

- (b) $[x - \bar{l}_n(\beta_i), x - l_n(\beta_i)] \cup [x + l_n(\beta_i), x + \bar{l}_n(\beta_i)] \subset \tilde{J}_{n,i}(s)^c$.

Proof. (a) Assume $\tilde{x} \in [x + l_n(\beta_i), x + \bar{l}_n(\beta_i)]$. Then

$$(3.16) \quad F_n(s, \tilde{x}) - F_n(s, x) = \int_{-\sqrt{a_n}}^{\sqrt{a_n}} \Phi^{m_{n+1}}(z) (u(s, \tilde{x} + z) - u(s, x + z)) dz.$$

For $|z| \leq \sqrt{a_n}$, let $x'' = \tilde{x} + z$ and $x' = x + z$. Then $|x' - x| \leq \sqrt{a_n}$ and

$$x'' - x' = \tilde{x} - x \in [l_n(\beta_i), \bar{l}_n(\beta_i)].$$

Therefore Lemma 3.4(b) and (3.16) imply

$$\begin{aligned} F_n(s, \tilde{x}) - F_n(s, x) &\geq \int_{-\sqrt{a_n}}^{\sqrt{a_n}} \Phi^{m_{n+1}}(z) 2^{-5} a_n^{\beta_i+1} (\tilde{x} - x) dz \\ &= 2^{-5} a_n^{\beta_i+1} (\tilde{x} - x). \end{aligned}$$

The proof for $\tilde{x} < x$ is similar.

(b) If $\tilde{x} \in [x - \bar{l}_n(\beta_i), x - l_n(\beta_i)] \cup [x + l_n(\beta_i), x + \bar{l}_n(\beta_i)]$, then

$$\begin{aligned} |F_n(s, \tilde{x})| &\geq |F_n(s, \tilde{x}) - F_n(s, x)| - |F_n(s, x)| \\ &\geq 2^{-5} a_n^{\beta_i+1} l_n(\beta_i) - a_n \quad (\text{by (a) and } (s, x) \in \tilde{J}_{n,i}) \\ &\geq \frac{33}{32} a_n. \end{aligned}$$

Therefore $\tilde{x} \notin \tilde{J}_{n,i}(s)$. ■

To ensure that (b) is not vacuous we obtain some crude lower bounds on the interval given there.

Lemma 3.6 *If $i \in \{0, \dots, L\}$, then*

$$l_n(\beta_i) < \sqrt{a_n} < \frac{1}{2} \bar{l}_n(\beta_i).$$

Proof.

$$\begin{aligned} l_n(\beta_i) a_n^{-1/2} &= (65 a_n^{\frac{1}{2} - \beta_i + 1}) \vee a_n^{\gamma - \frac{1}{2} - 2\beta_i(1-\gamma) - \varepsilon_1} \\ &\leq (65 a_n^{\varepsilon_1}) \vee a_n^{2\gamma - \frac{3}{2} - \varepsilon_1} \quad (\text{since } \beta_i < 1/2) \\ &< 1 \end{aligned}$$

by (3.10) and because $a_n^{\varepsilon_1} < 2^{-8}$ by (3.14). This gives the first inequality. For the second one, use $\beta_i \leq \frac{1}{2} - 6\varepsilon_1$ and (3.14) to see that

$$\sqrt{a_n} \bar{l}_n(\beta_i)^{-1} = a_n^{\frac{1}{2} - \beta_i - 5\varepsilon_1} \leq a_n^{\varepsilon_1} < 1/2. \quad \blacksquare$$

Let $|A|$ denote the Lebesgue measure of $A \subset \mathbb{R}$.

Lemma 3.7 *For all $i \in \{0, \dots, L-1\}$ and $s \geq 0$,*

$$|\tilde{J}_{n,i}(s)| \leq 10K_0 \bar{l}_n(\beta_i)^{-1} l_n(\beta_i).$$

Proof. Fix s, i as above. Let $\mathcal{I}_{n,i}(x) = (x - l_n(\beta_i), x + l_n(\beta_i)) \subset \tilde{\mathcal{I}}_{n,i}(x) = (x - \bar{l}_n(\beta_i), x + \bar{l}_n(\beta_i))$. This inclusion follows from Lemma 3.6. The compactness of $\tilde{\mathcal{J}}_{n,i}(s)$ (Proposition 3.3) implies there are $x_1, \dots, x_Q \in \tilde{\mathcal{J}}_{n,i}(s)$ so that $\tilde{\mathcal{J}}_{n,i}(s) \subset \bigcup_{j=1}^Q \mathcal{I}_{n,i}(x_j)$.

Assume that for some $k \neq j$, $|x_k - x_j| \leq \bar{l}_n(\beta_i)/2$. We claim that $\mathcal{I}_{n,i}(x_j) \subset \tilde{\mathcal{I}}_{n,i}(x_k)$. Indeed, if $y \in \mathcal{I}_{n,i}(x_j)$ then

$$|y - x_k| \leq |y - x_j| + |x_j - x_k| < l_n(\beta_i) + \bar{l}_n(\beta_i)/2 < \bar{l}_n(\beta_i),$$

the last by Lemma 3.6, and the claim is proved. Lemma 3.5(b) implies

$$\tilde{\mathcal{J}}_{n,i}(s) \cap (\tilde{\mathcal{I}}_{n,i}(x_k) - \mathcal{I}_{n,i}(x_k)) = \emptyset,$$

and so the above claim gives

$$\mathcal{I}_{n,i}(x_j) \cap \tilde{\mathcal{J}}_{n,i}(s) \subset \tilde{\mathcal{I}}_{n,i}(x_k) \cap \tilde{\mathcal{J}}_{n,i}(s) = \mathcal{I}_{n,i}(x_k) \cap \tilde{\mathcal{J}}_{n,i}(s).$$

Therefore we may omit $\mathcal{I}_{n,i}(x_j)$ and still have a cover of $\tilde{\mathcal{J}}_{n,i}(s)$. Doing this sequentially for x_1, \dots, x_Q , we may therefore assume that

$$|x_k - x_j| > \bar{l}_n(\beta_i)/2 \quad \text{for all } k \neq j.$$

Since each $x_j \in \tilde{\mathcal{J}}_{n,i}(s) \subset [-K_0, K_0]$, this implies $Q \leq 2K_0(\bar{l}_n(\beta_i)/2)^{-1} + 1$, and therefore

$$|\tilde{\mathcal{J}}_{n,i}(s)| \leq (4K_0\bar{l}_n(\beta_i)^{-1} + 1)2l_n(\beta_i) \leq 10K_0\bar{l}_n(\beta_i)^{-1}l_n(\beta_i).$$

■

Proof of (H_2) in Proposition 2.1. Fix $M \in \mathbb{N}$. Recall from our discussion after the definition of $J_{n,i}$ sets that it suffices to show that for all $i \in \{0, \dots, L\}$,

$$(H_{2,i}) \quad \lim_{n \rightarrow \infty} E(I_i^n(t_0 \wedge U_{M,n})) = 0.$$

We will in fact show that if we strengthen (3.14) to

$$(3.17) \quad n > n_M(\varepsilon_1) \vee n_0(\varepsilon_0, \varepsilon_1) \vee \frac{2}{\varepsilon_1},$$

then we have the stronger \mathbb{L}^∞ bound

$$(3.18) \quad I_i^n(t_0 \wedge U_{M,n}) \leq c_1(\Psi)t_0K_0a_n^{\gamma - \frac{3}{4}},$$

which clearly implies $(H_{2,i})$ since $\gamma > \frac{3}{4}$. Proposition 3.3, $\text{Supp}(\Phi_x^{m_{n+1}}) \subset [x - \sqrt{a_n}, x + \sqrt{a_n}]$ and $\frac{2}{n} < \varepsilon_1$ (by (3.17)) imply

$$(3.19) \quad \begin{aligned} I_i^n(t_0 \wedge U_{M,n}) &\leq a_n^{-\frac{3}{2} - \varepsilon_1} \int_0^{t_0} \int \int 1(s < U_{M,n}) 1_{\tilde{\mathcal{J}}_{n,i}(s)}(x) |u(s, y)|^{2\gamma} \\ &\quad \times 1(|y - x| \leq \sqrt{a_n}) \Phi_x^{m_{n+1}}(y) \Psi_s(x) dy dx ds. \end{aligned}$$

Consider first (3.18) for $i = 0$. For $x \in \tilde{J}_{n,0}(s)$ and $|y - x| \leq \sqrt{a_n}$, we have $|u(s, y)| \leq 3a_n^{(1-\varepsilon_0)/2}$ and so from (3.19),

$$\begin{aligned} I_0^n(t_0 \wedge U_{M,n}) &\leq a_n^{-\frac{3}{2}-\varepsilon_1} 3^{2\gamma} a_n^{\gamma(1-\varepsilon_0)} \|\Psi\|_\infty \int_0^{t_0} |\tilde{J}_{n,0}(s)| ds \\ &\leq a_n^{-\frac{3}{2}-\varepsilon_1} 3^{2\gamma} a_n^{\gamma(1-\varepsilon_0)} \|\Psi\|_\infty t_0 10K_0 a_n^{-5\varepsilon_1} ((65a_n^{1-\varepsilon_0}) \vee a_n^{\gamma-\varepsilon_1}) \quad (\text{by Lemma 3.7}) \\ &\leq c_1(\Psi) t_0 K_0 a_n^{2\gamma-\frac{3}{2}} a_n^{-\gamma\varepsilon_0-7\varepsilon_1} \\ &\leq c_1(\Psi) t_0 K_0 a_n^{\gamma-\frac{3}{4}}, \end{aligned}$$

as required, where (3.10) is used in the last two lines.

Consider now $i \in \{1, \dots, L\}$. Assume $x \in \tilde{J}_{n,i}(s)$ and $|y - x| \leq \sqrt{a_n}$. We have $|\langle u_s, \Phi_x^{m_{n+1}} \rangle| \leq a_n$ and

$$\text{Supp}(\Phi_x^{m_{n+1}}) \subset [x - \sqrt{a_n}, x + \sqrt{a_n}].$$

Using the continuity of $u(s, \cdot)$, we conclude that

$$|u(s, \hat{x}_n(s, x))| \leq a_n,$$

and of course we have $|\hat{x}_n(s, x) - x| \leq \sqrt{a_n}$. Therefore

$$(3.20) \quad \begin{aligned} |y - \hat{x}_n(s, x)| &\leq 2\sqrt{a_n} \\ &\leq \bar{l}_n(\beta_i) \quad (\text{by Lemma 3.6}). \end{aligned}$$

Apply Lemma 3.4(a) with $x'' = y$ and $x' = \hat{x}_n(s, x)$, to see that

$$(3.21) \quad \begin{aligned} |u(s, y)| &\leq |u(s, \hat{x}_n(s, x))| + 2a_n^{\beta_i} (|y - \hat{x}_n(s, x)| \vee a_n^{\gamma-2\beta_i(1-\gamma)-\varepsilon_1}) \\ &\leq a_n + 4a_n^{\beta_i+\frac{1}{2}} \leq 5a_n^{\beta_i+\frac{1}{2}}, \end{aligned}$$

where (3.20) and Lemma 3.6 are used in the next to last inequality. Use (3.21) in (3.19) and conclude that

$$(3.22) \quad I_i^n(t_0 \wedge U_{M,n}) \leq a_n^{-\frac{3}{2}-\varepsilon_1} 5^{2\gamma} a_n^{2\gamma(\beta_i+\frac{1}{2})} \|\Psi\|_\infty \int_0^{t_0} |\tilde{J}_{n,i}(s)| ds, \quad i = 1, \dots, L.$$

Assume now $1 \leq i \leq L - 1$. Apply Lemma 3.7 to the right-hand side of (3.22) to see that

$$(3.23) \quad \begin{aligned} I_i^n(t_0 \wedge U_{M,n}) &\leq c_1(\Psi) t_0 K_0 a_n^{-\frac{3}{2}-\varepsilon_1+2\gamma(\beta_i+\frac{1}{2})} (a_n^{1-\beta_{i+1}} \vee a_n^{\gamma-2\beta_i(1-\gamma)-\varepsilon_1}) a_n^{-5\varepsilon_1-\beta_i} \\ &= c_1(\Psi) t_0 K_0 [a_n^{\rho_{1,i}} \vee a_n^{\rho_{2,i}}]. \end{aligned}$$

A bit of arithmetic shows that

$$\begin{aligned} \rho_{1,i} &= \gamma - \frac{1}{2} - 2\beta_i(1-\gamma) - \varepsilon_0 - 6\varepsilon_1 \\ &> 2\gamma - \frac{3}{2} - \varepsilon_0 - 6\varepsilon_1 \quad (\text{use } \beta_i < 1/2) \\ &> \gamma - \frac{3}{4}, \end{aligned}$$

where (3.10) is used in the last inequality. We also have

$$\begin{aligned}\rho_{2,i} &= -\frac{3}{2} + 2\gamma + \beta_i(4\gamma - 3) - 7\varepsilon_1 \\ &> 2\gamma - \frac{3}{2} - 7\varepsilon_1 > \gamma - \frac{3}{4},\end{aligned}$$

again using (3.10) in the last inequality. Use these bounds on $\rho_{l,i}$, $l = 1, 2$ in (3.23) to prove (3.18) for $i \leq i \leq L - 1$.

It remains to prove (3.18) for $i = L$. For this, use the trivial bound $|\tilde{J}_{n,i}(s)| \leq 2K_0$ in (3.22) and obtain

$$\begin{aligned}I_L^n(t_0 \wedge U_{M,n}) &\leq a_n^{-\frac{3}{2}-\varepsilon_1} 5^{2\gamma} a_n^{2\gamma(\beta_L + \frac{1}{2})} \|\Psi\|_\infty 2K_0 t_0 \\ &\leq c_1(\Psi) K_0 t_0 a_n^{-\frac{3}{2}-\varepsilon_1 + \gamma + 2\gamma(\frac{1}{2} - 6\varepsilon_1 - \varepsilon_0)} \\ &\leq c_1(\Psi) K_0 t_0 a_n^{2\gamma - \frac{3}{2} - 15\varepsilon_1} \\ &\leq c_1(\Psi) K_0 t_0 a_n^{\gamma - \frac{3}{4}},\end{aligned}$$

yet again using (3.10) in the last. This proves (3.18) in the last case of $i = L$. Having proved (3.18) in all cases, we have finished the proof of (H_2) . \blacksquare

Proposition 2.1 therefore applies and establishes Theorem 1.2 for $b \equiv 0$, except for the proof of Proposition 3.3. This will be the objective of the next three sections.

4 Some Integral Bounds for Heat Kernels

If $0 < p \leq 1$, $q \in \mathbb{R}$ and $0 \leq \Delta_2 \leq \Delta_1 \leq t$, define

$$J_{p,q}(\Delta_1, \Delta_2, \Delta) = \int_{t-\Delta_1}^{t-\Delta_2} (t-s)^q \left(1 \wedge \frac{\Delta}{t-s}\right)^p ds.$$

These integrals will arise frequently in our modulus of continuity estimates.

Lemma 4.1 (a) *If $q > p - 1$, then*

$$(4.1) \quad J_{p,q}(\Delta_1, \Delta_2, \Delta) \leq \frac{2}{q+1-p} (\Delta \wedge \Delta_1)^p \Delta_1^{q+1-p}.$$

(b) *If $-1 < q < p - 1$, then*

$$(4.2) \quad \begin{aligned} &J_{p,q}(\Delta_1, \Delta_2, \Delta) \\ &\leq ((p-1-q)^{-1} + (q+1)^{-1}) [(\Delta \wedge \Delta_1)^{q+1} 1(\Delta_2 \leq \Delta) + (\Delta \wedge \Delta_1)^p \Delta_2^{q-p+1} 1(\Delta_2 > \Delta)] \end{aligned}$$

$$(4.3) \quad \leq ((p-1-q)^{-1} + (q+1)^{-1}) \Delta^p (\Delta \vee \Delta_2)^{q-p+1}.$$

(c) *If $q < -1$, then*

$$(4.4) \quad J_{p,q}(\Delta_1, \Delta_2, \Delta) \leq 2|q+1|^{-1} (\Delta \wedge \Delta_2)^p \Delta_2^{q+1-p}.$$

Proof. For all p, q as above,

$$\begin{aligned}
J_{p,q} &= \int_{\Delta_2}^{\Delta_1} u^q \left(1 \wedge \frac{\Delta}{u}\right)^p du \\
(4.5) \quad &= 1(\Delta_2 < \Delta) \int_{\Delta_2}^{\Delta \wedge \Delta_1} u^q du + 1(\Delta_1 > \Delta) \int_{\Delta_2 \vee \Delta}^{\Delta_1} \Delta^p u^{q-p} du.
\end{aligned}$$

(a) From (4.5),

$$\begin{aligned}
J_{p,q} &\leq 1(\Delta_2 < \Delta) \frac{(\Delta \wedge \Delta_1)^{q+1}}{q+1} + 1(\Delta_1 > \Delta) \Delta^p \frac{\Delta_1^{q-p+1}}{q-p+1} \\
&\leq ((q+1)^{-1} + (q-p+1)^{-1})(\Delta \wedge \Delta_1)^p \Delta_1^{q+1-p},
\end{aligned}$$

which gives the required bound.

(b) Again (4.5) implies

$$\begin{aligned}
J_{p,q} &\leq 1(\Delta_2 < \Delta) \frac{(\Delta \wedge \Delta_1)^{q+1}}{q+1} + 1(\Delta_1 > \Delta) \Delta^p \frac{(\Delta_2 \vee \Delta)^{q+1-p}}{p-1-q} \\
&\leq 1(\Delta_2 \leq \Delta) (\Delta \wedge \Delta_1)^{q+1} ((q+1)^{-1} + (p-1-q)^{-1}) \\
&\quad + 1(\Delta_2 > \Delta) (\Delta \wedge \Delta_1)^p \Delta_2^{q-p+1} (p-1-q)^{-1},
\end{aligned}$$

which gives the first inequality. The second inequality is elementary.

(c) By (4.5),

$$\begin{aligned}
J_{p,q} &\leq 1(\Delta_2 < \Delta) \frac{\Delta_2^{q+1}}{|q+1|} + 1(\Delta_1 > \Delta) \Delta^p \frac{(\Delta_2 \vee \Delta)^{q+1-p}}{p-1-q} \\
&\leq \frac{1(\Delta_2 < \Delta)}{|q+1|} (\Delta_2 \wedge \Delta)^p \Delta_2^{q+1-p} + 1(\Delta_2 \leq \Delta < \Delta_1) \frac{\Delta^{q+1}}{p-1-q} \\
&\quad + 1(\Delta < \Delta_2) \frac{(\Delta \wedge \Delta_2)^p \Delta_2^{q+1-p}}{p-1-q} \\
&\leq \frac{2}{|q+1|} (\Delta \wedge \Delta_2)^p \Delta_2^{q+1-p},
\end{aligned}$$

where we used $\Delta^{q+1} \leq \Delta_2^{q+1} = (\Delta \wedge \Delta_2)^p \Delta_2^{q+1-p}$ if $\Delta_2 \leq \Delta$, and $|q+1|^{-1} \geq (p-1-q)^{-1}$ in the last line. \blacksquare

We let $p'_t(x) = \frac{d}{dx} p_t(x)$.

Lemma 4.2

$$|p'_t(z)| \leq c_{4.2} t^{-1/2} p_{2t}(z).$$

Proof. Trivial. ■

Lemma 4.3 (a) *There is a $c_{4.3}$ so that for any $s < t \leq t', x, x' \in \mathbb{R}$,*

$$(4.6) \quad \int (p_{t'-s}(y-x') - p_{t-s}(y-x))^2 dy \leq c_{4.3}(t-s)^{-1/2} \left[1 \wedge \frac{d((t,x), (t',x'))^2}{t-s} \right].$$

(b) *For any $R > 2$ there is a $c_{4.3}(R)$ so that for any $0 \leq p, r \leq R$, $\eta_0, \eta_1 \in (1/R, 1/2)$, $0 \leq s < t \leq t' \leq R$, $x, x' \in \mathbb{R}$,*

$$(4.7) \quad \int e^{r|y-x|} |y-x|^p (p_{t-s}(y-x) - p_{t'-s}(y-x'))^2 1(|y-x| > (t-s)^{1/2-\eta_0} \vee 2|x'-x|) dy \\ \leq c_{4.3}(R)(t-s)^{-1/2} \exp\{-\eta_1(t-s)^{-2\eta_0}/33\} \left[1 \wedge \frac{d((t,x), (t',x'))^2}{t-s} \right]^{1-(\eta_1/2)}.$$

Proof. (a) Let $f(u) = u^{-1/2}$. By Chapman-Kolmogorov, the integral in (4.6) equals

$$(4.8) \quad p_{2(t'-s)}(0) + p_{2(t-s)}(0) - 2p_{t'-s+t-s}(0) + 2(p_{t'-s+t-s}(0) - p_{t'-s+t-s}(x-x')) \\ \leq (2\pi)^{-1/2} \left[|f(2(t'-s)) + f(2(t-s)) - 2f(t'-s+t-s)| \right. \\ \left. + (t'-s+t-s)^{-1/2} \left(1 - \exp\left\{ \frac{-(x-x')^2}{2(t'-s+t-s)} \right\} \right) \right] \\ \equiv (2\pi)^{-1/2} [T_1 + T_2].$$

Clearly T_2 is at most $(t'-s)^{-1/2} \left[1 \wedge \frac{|x-x'|^2}{t'-s} \right]$. If $0 < u \leq u'$, $0 \leq f(u) - f(u') \leq u^{-1/2} \wedge [u^{-3/2}|u' - u|]$ (by the Mean Value Theorem) and so

$$(4.9) \quad T_1 \leq \left[\sqrt{2}(t-s)^{-1/2} \wedge \left(\frac{2}{(2(t-s))^{3/2}} |t' - t| \right) \right]$$

Use the above bounds on T_1 and T_2 in (4.8) to complete the proof of (a).

(b) This proof is very similar to that of Lemma 4.4 (b) below and so is omitted. There are some minor differences leading to the factor of 1/33 (rather than the 1/64 in Lemma 4.4 (b))—e.g., the much simpler analogue of (4.20) below has $p_u(w)$ on the right side. ■

Lemma 4.4 (a) *There is a $c_{4.4}$ so that for any $s < t \leq t', x, x' \in \mathbb{R}$,*

$$(4.10) \quad \int (p'_{t'-s}(y-x') - p'_{t-s}(y-x))^2 dy \leq c_{4.4}(t-s)^{-3/2} \left[1 \wedge \frac{d((t,x), (t',x'))^2}{t-s} \right].$$

(b) *For any $R > 2$ there is a $c_{4.4}(R)$ so that for any $0 \leq p, r \leq R$, $\eta_0, \eta_1 \in (1/R, 1/2)$, $0 \leq s < t \leq t' \leq R$, $x, x' \in \mathbb{R}$,*

$$(4.11) \quad \int e^{r|y-x|} |y-x|^p (p'_{t-s}(y-x) - p'_{t'-s}(y-x'))^2 1(|y-x| > (t-s)^{1/2-\eta_0} \vee 2|x'-x|) dy \\ \leq c_{4.4}(R)(t-s)^{-3/2} \exp\{-\eta_1(t-s)^{-2\eta_0}/64\} \left[1 \wedge \frac{d((t,x), (t',x'))^2}{t-s} \right]^{1-(\eta_1/2)}.$$

Proof. (a) We first claim that

$$(4.12) \quad \int p'_t(w)p'_t(w-x)dw = \left(\frac{t}{2} - \frac{x^2}{4}\right)\frac{p_{2t}(x)}{t^2}.$$

To see this, first do a bit of algebra to get

$$(4.13) \quad p_t(w)p_t(w-x) = p_{t/2}(w-(x/2))p_{2t}(x).$$

Therefore the left-hand side of (4.12) equals

$$\begin{aligned} \int \frac{w(w-x)}{t^2}p_t(w)p_t(w-x)dw &= \int \frac{w(w-x)}{t^2}p_{t/2}(w-(x/2))dw p_{2t}(x) \\ &= \int \left(u^2 - \frac{x^2}{4}\right)p_{t/2}(u)du \frac{p_{2t}(x)}{t^2} \quad (u = w - (x/2)) \\ &= \left(\frac{t}{2} - \frac{x^2}{4}\right)\frac{p_{2t}(x)}{t^2}, \end{aligned}$$

giving the right-hand side of (4.12).

Next we claim that

$$(4.14) \quad \int p'_{t'}(w-x)p'_t(w-x)dw = (t+t')^{-1}p_{t'+t}(0).$$

Some algebra shows that

$$(4.15) \quad p_{t'}(w)p_t(w) = p_{t'+t}(0)p_{\frac{t't}{t+t'}}(w).$$

Therefore the left-hand side of (4.14) equals

$$\int \frac{w^2}{t't}p_{t'}(w)p_t(w)dw = \int \frac{w^2}{t't}p_{\frac{t't}{t+t'}}(w)dw p_{t'+t}(0) = (t+t')^{-1}p_{t'+t}(0),$$

and we have (4.14).

The left-hand side of (4.10) is bounded by

$$(4.16) \quad 2\left[\int (p'_{t'-s}(y-x') - p'_{t'-s}(y-x))^2 dy + \int (p'_{t'-s}(y-x) - p'_{t-s}(y-x))^2 dy\right] \\ \equiv 2(T_1 + T_2).$$

Now expand T_1 and use (4.12) to see that

$$\begin{aligned} T_1 &= 2 \int p'_{t'-s}(y-x)^2 dy - 2 \int p'_{t'-s}(w)p'_{t'-s}(w+x-x')dw \\ &= (t'-s)\frac{p_{2(t'-s)}(0)}{(t'-s)^2} - \left((t'-s) - \frac{(x-x')^2}{2}\right)\frac{p_{2(t'-s)}(x'-x)}{(t'-s)^2} \\ &= (t'-s)^{-1}(p_{2(t'-s)}(0) - p_{2(t'-s)}(x'-x)) + \frac{(x'-x)^2}{2(t'-s)^2}p_{2(t'-s)}(x'-x) \\ &\leq c_0(t'-s)^{-3/2}\left[1 \wedge \frac{(x'-x)^2}{t'-s}\right]. \end{aligned}$$

Finally let $g(u) = u^{-3/2}$, and expand T_2 and use (4.14) to conclude

$$\begin{aligned} T_2 &= (2(t' - s))^{-1} p_{2(t'-s)}(0) + (2(t - s))^{-1} p_{2(t-s)}(0) - 2(t - s + t' - s)^{-1} p_{t-s+t'-s}(0) \\ &= (2\pi)^{-1/2} [g(2(t' - s)) + g(2(t - s)) - 2g(t' - s + t - s)] \\ &\leq (2\pi)^{-1/2} 2 \left[(2(t - s))^{-3/2} \wedge (2(t - s))^{-5/2} |t' - t| \right]. \end{aligned}$$

The last inequality follows as in (4.9). Use the above bounds on T_1 and T_2 in (4.16) to complete the proof of (a).

(b) Note that $|y - x| > (t' - s)^{1/2 - \eta_0} \vee 2|x' - x|$ implies that

$$(4.17) \quad |y - x'| \geq |y - x| - |x' - x| \geq |y - x|/2 \geq \frac{(t' - s)^{1/2 - \eta_0}}{2}$$

and in particular from the second inequality,

$$(4.18) \quad |y - x| \leq 2|y - x'|.$$

Assume p, r, η_i, s, t, t' as in (b) and let $d = d((t, x), (t', x'))$. By Hölder's inequality and then (a), the integral on the left-hand side of (4.11) is at most

$$\begin{aligned} & \left[\int (p'_{t-s}(y - x) - p'_{t'-s}(y - x'))^2 dy \right]^{1 - \eta_1/2} \\ & \times \left[\int e^{\frac{2r}{\eta_1}|y-x|} |y - x|^{2p/\eta_1} (p'_{t-s}(y - x) - p'_{t'-s}(y - x'))^2 \mathbf{1}(|y - x| > (t' - s)^{1/2 - \eta_0} \vee 2|x - x'|) dy \right]^{\eta_1/2} \\ & \leq c_1 (t - s)^{-\frac{3}{2} + \frac{3\eta_1}{4}} \left[1 \wedge \frac{d^2}{t - s} \right]^{1 - \frac{\eta_1}{2}} \left[\int e^{\frac{2r}{\eta_1}|y-x|} |y - x|^{2p/\eta_1} (p'_{t-s}(y - x)^2 + p'_{t'-s}(y - x')^2) \right. \\ & \quad \left. \times \mathbf{1}(|y - x| > (t' - s)^{1/2 - \eta_0} \vee 2|x - x'|) dy \right]^{\eta_1/2} \\ & \leq c_2(R) (t - s)^{-\frac{3}{2} + \frac{3\eta_1}{4}} \left[1 \wedge \frac{d^2}{t - s} \right]^{1 - \frac{\eta_1}{2}} \left[\int e^{\frac{2r}{\eta_1}|w|} |w|^{\frac{2p}{\eta_1}} p'_{t-s}(w)^2 \mathbf{1}(|w| > (t - s)^{1/2 - \eta_0}) dw \right. \\ (4.19) \quad & \quad \left. + \int e^{\frac{4r}{\eta_1}|w|} |w|^{\frac{2p}{\eta_1}} p'_{t'-s}(w)^2 \mathbf{1}(|w| > (t' - s)^{1/2 - \eta_0}/2) dw \right]^{\eta_1/2}, \end{aligned}$$

where in the last we used (4.17) and (4.18).

If $|w| > \frac{1}{2}u^{1/2 - \eta_0}$, then by Lemma 4.2,

$$(4.20) \quad \begin{aligned} p'_u(w)^2 &\leq c_{4.2}^2 u^{-1} p_{2u}(w)^2 \leq c_{4.2}^2 u^{-3/2} e^{-u^{-2\eta_0}/16} p_{2u}(w) \\ &\leq c_3(R) p_{2u}(w). \end{aligned}$$

The fact that $\eta_0 > 1/R$ is used in the last line. Use the above to show that

$$\begin{aligned} & \left[\int e^{\frac{4r}{\eta_1}|w|} |w|^{\frac{2p}{\eta_1}} p'_{t'-s}(w)^2 \mathbf{1}(|w| > (t' - s)^{1/2 - \eta_0}/2) dw \right]^{\eta_1/2} \\ & \leq c_4(R) \left[\int e^{\frac{4r}{\eta_1}|w|} |w|^{\frac{2p}{\eta_1}} p_{2(t'-s)}(w) \mathbf{1}(|w| > (t' - s)^{1/2 - \eta_0}/2) dw \right]^{\eta_1/2} \\ & \leq c_5(R) E(e^{\frac{8r}{\eta_1} \sqrt{2(t'-s)} |B_1|} |_{B_1}^{\frac{4p}{\eta_1}})^{\eta_1/4} P(|B_1| > \frac{1}{2\sqrt{2}}(t' - s)^{-\eta_0})^{\eta_1/4} \quad (\text{by Hölder}) \\ & \leq c_6(R) \exp\{-\eta_1(t' - s)^{-2\eta_0}/64\}. \end{aligned}$$

In the last line the Brownian integral is bounded by some $c(R)$ thanks to the bounds on the constants involved. Use the same bound with t in place of t' to see that the right-hand side of (4.19), and hence of (4.11), is at most

$$c_7(R)(t-s)^{3\eta_1/4}(t-s)^{-3/2}\left[1 \wedge \frac{d^2}{t-s}\right]^{1-\frac{\eta_1}{2}} \exp\{-(t'-s)^{-2\eta_0}\eta_1/64\}.$$

The result follows because $(t-s)^{3\eta_1/4} \leq R$. ■

5 Local Bounds on the Difference of Two Solutions

This section is devoted to establishing the local bounds on the difference of two solutions to (1.5). These bounds are crucial for the construction of the stopping times in Proposition 3.3, which is then carried out in Section 6. We continue to assume throughout this Section that $b \equiv 0$. Recall that X^1, X^2 are two solutions as in Section 2, $u = X^1 - X^2$, and we assume the hypotheses of Theorem 1.2 as well as (2.29).

We refine the earlier set $Z(N, K)$ and define, for $K, N, n \in \mathbb{N}$ and $\beta \in (0, 1/2]$,

$$Z(N, n, K, \beta)(\omega) = \{(t, x) \in [0, T_K] \times [-K, K] : \text{there is a } (\hat{t}_0, \hat{x}_0) \in [0, T_K] \times \mathbb{R} \text{ such that} \\ d((\hat{t}_0, \hat{x}_0), (t, x)) \leq 2^{-N}, |u(\hat{t}_0, \hat{x}_0)| \leq a_n \wedge (\sqrt{a_n}2^{-N}), \text{ and } |u'_{1, a_n}(\hat{t}_0, \hat{x}_0)| \leq a_n^\beta\},$$

and for $\beta = 0$ define $Z(N, n, K, 0)(\omega) = Z(N, n, K)(\omega)$ as above, but with the condition on $|u'_{1, a_n}(\hat{t}_0, \hat{x}_0)|$ omitted.

Recalling $\frac{3}{4} < \gamma < 1$, we let

$$(5.1) \quad \gamma_m = \frac{(\gamma - 1/2)(1 - \gamma^m)}{1 - \gamma} + 1, \quad \tilde{\gamma}_m = \gamma_m \wedge 2$$

so that we have the recursion relation

$$(5.2) \quad \gamma_{m+1} = \gamma\gamma_m + 1/2, \quad \gamma_0 = 1.$$

Clearly γ_m increases to $\gamma_\infty = (\gamma - 1/2)(1 - \gamma)^{-1} + 1 = (2(1 - \gamma))^{-1} > 2$ and so we may define a finite natural number, $\bar{m} > 1$, by

$$(5.3) \quad \bar{m} = \min\{m : \gamma_{m+1} > 2\} = \min\{m : \gamma\gamma_m > 3/2\}.$$

Note that

$$(5.4) \quad \tilde{\gamma}_{\bar{m}+1} = 2.$$

Definition. A collection of $[0, \infty]$ -valued random variables, $\{N(\alpha) : \alpha \in A\}$, is stochastically bounded uniformly in α iff

$$\lim_{M \rightarrow \infty} \sup_{\alpha \in A} P(N(\alpha) \geq M) = 0.$$

Finally we introduce the condition whose proof will be the goal of this section. Recall that K_1 is as in (2.36).

Property (P_m) . For $m \in \mathbb{Z}_+$ we will let (P_m) denote the following property:

$$(5.5) \quad \begin{aligned} & \text{For any } n \in \mathbb{N}, \xi, \varepsilon_0 \in (0, 1), K \in \mathbb{N}^{\geq K_1} \text{ and } \beta \in [0, 1/2], \text{ there is an} \\ & N_1(\omega) = N_1(m, n, \xi, \varepsilon_0, K, \beta) \text{ in } \mathbb{N} \text{ a.s. such that for all } N \geq N_1, \\ & \text{if } (t, x) \in Z(N, n, K, \beta), t' \leq T_K, \text{ and } d((t, x), (t', x')) \leq 2^{-N}, \\ & \text{then } |u(t', x')| \leq a_n^{-\varepsilon_0} 2^{-N\xi} \left[(\sqrt{a_n} \vee 2^{-N})^{\tilde{\gamma}_m - 1} + a_n^\beta \mathbf{1}(m > 0) \right]. \\ & \text{Moreover } N_1 \text{ is stochastically bounded uniformly in } (n, \beta). \end{aligned}$$

Here is the main result of this section:

Proposition 5.1 *For any $m \leq \bar{m} + 1$, (P_m) holds.*

Proof. (P_0) is an easy consequence of Theorem 2.3, as we now show—and we may even take $\varepsilon_0 = 0$. Let $\xi \in (0, 1)$ and apply Theorem 2.3 with $\xi' = (\xi + 1)/2$ in place of ξ . If $(t, x) \in Z(N, K) \cap Z(N, n, K, \beta)$ and (\hat{t}_0, \hat{x}_0) is as in the definition of $Z(N, K)$, then $(\hat{t}_0, \hat{x}_0) \in Z(N, K + 1)$ (we need $K + 1$ since $|\hat{x}_0| \leq K + 1$). Theorem 2.3 implies that if $N \geq N_0(\xi', K + 1) \vee 4(1 - \xi)^{-1} \equiv N_1(0, \xi, K)$ (it doesn't depend on (n, β) and there is no ε_0), then

$$|u(t, x)| \leq 2^{-N\xi'} + |u(\hat{t}_0, \hat{x}_0)| \leq 2^{1-N\xi'}.$$

If (t', x') is as in (P_0) , the above and Theorem 2.3 imply

$$|u(t', x')| \leq |u(t, x)| + |u(t', x') - u(t, x)| \leq 2^{1-N\xi'} + 2^{-N\xi'} \leq 2^{2-N\xi'} \leq 2^{-N\xi},$$

where the last inequality holds because $N \geq 4(1 - \xi)^{-1}$. (P_0) follows.

The induction step will require some additional continuity results which also will be used directly in the next section. We start by noting that (P_m) easily gives some global bounds on $|u|$.

Lemma 5.2 *Let $0 \leq m \leq \bar{m} + 1$ and assume (P_m) . For any n, ξ, ε_0, K and β as in (P_m) , if $\bar{d}_N = 2^{-N} \vee d((s, y), (t, x))$ and $\sqrt{C_{5.2}(\omega)} = (4a_n^{-\varepsilon_0} + 2^{2N_1(\omega)} 2K e^K)$, then for any $N \in \mathbb{N}$,*

$$(5.6) \quad \text{on } \{\omega : N \geq N_1(m, n, \xi, \varepsilon_0, K, \beta), (t, x) \in Z(N, n, K, \beta)\},$$

we have

$$(5.7) \quad \begin{aligned} |u(s, y)| & \leq \sqrt{C_{5.2}} e^{|y-x|} \bar{d}_N^\xi \\ & \times \left[(\sqrt{a_n} \vee \bar{d}_N)^{\tilde{\gamma}_m - 1} + \mathbf{1}(m > 0) a_n^\beta \right] \text{ for all } s \leq T_K \text{ and } y \in \mathbb{R}. \end{aligned}$$

Proof. Assume N, ω, t, x are as in (5.6).

Case 1. $d \equiv d((s, y), (t, x)) \leq 2^{-N_1}$.

If $d > 2^{-N}$ choose $N_1 \leq N' < N$ so that $2^{-N'-1} < d \leq 2^{-N'}$, and if $d \leq 2^{-N}$ set $N' = N$. Then $(t, x) \in Z(N', n, K, \beta)$, $d \leq 2^{-N'} \leq 2^{-N} \vee 2d \leq 2\bar{d}_N$ and so by (P_m) for $s \leq T_K$,

$$\begin{aligned} |u(s, y)| & \leq a_n^{-\varepsilon_0} 2^{-N'\xi} \left[(\sqrt{a_n} \vee 2^{-N'})^{\tilde{\gamma}_m - 1} + \mathbf{1}(m > 0) a_n^\beta \right] \\ & \leq 4a_n^{-\varepsilon_0} (\bar{d}_N)^\xi \left[(\sqrt{a_n} \vee \bar{d}_N)^{\tilde{\gamma}_m - 1} + \mathbf{1}(m > 0) a_n^\beta \right]. \end{aligned}$$

Case 2. $d > 2^{-N_1}$.

As $K \geq K_1$, for $s \leq T_K$,

$$\begin{aligned} |u(s, y)| &\leq 2K e^{|y|} \leq 2K e^{|y|} (d2^{N_1})^{\xi + \tilde{\gamma}m - 1} \\ &\leq 2K e^K e^{|y-x|} 2^{2N_1} (\bar{d}_N)^{\xi + \tilde{\gamma}m - 1}. \end{aligned}$$

The Lemma follows from the above two bounds. \blacksquare

Remark 5.3 If $m = 0$ we may set $\varepsilon_0 = 0$ in the above and N_1 will not depend on $(n, \varepsilon_0, \beta)$ by the above proof of (P_0) .

To carry out the induction we first use (P_m) to obtain a local modulus of continuity for F_δ . Recall F_δ is given by (3.6) where

$$D(r, y) = \sigma(r, y, X^1(r, y)) - \sigma(r, y, X^2(r, y)).$$

From the representation in (3.6), we have for $s \leq t \leq t'$ and $s' \leq t'$

$$\begin{aligned} (5.8) \quad |F_\delta(s, t, x) - F_\delta(s', t', x')| &\leq |F_\delta(s, t', x') - F_\delta(s', t', x')| + |F_\delta(s, t', x') - F_\delta(s, t, x)| \\ &= \left| \int_{(s-\delta)^+}^{(s'-\delta)^+} \int p'_{t'-r}(y-x') D(r, y) W(dr, dy) \right| \\ &\quad + \left| \int_0^{(s-\delta)^+} (p'_{t'-r}(y-x') - p'_{t-r}(y-x)) D(r, y) W(dr, dy) \right|. \end{aligned}$$

This decomposition and (3.2) suggest we introduce the following square functions for $\eta_0 \in (0, 1/2)$ and $\delta \in (0, 1]$, and $s \leq t \leq t', s' \leq t'$:

$$\begin{aligned} (5.9) \quad Q_{T,\delta}(s, s', t', x') &= \int_{(s \wedge s' - \delta)^+}^{(s \vee s' - \delta)^+} \int p'_{t'-r}(y-x')^2 e^{2R_1|y|} |u(r, y)|^{2\gamma} dy dr, \\ Q_{S,1,\delta,\eta_0}(s, t, x, t', x') &= \int_0^{(s-\delta)^+} \int 1(|y-x| > (t'-r)^{1/2-\eta_0} \vee 2|x-x'|) \\ &\quad \times (p'_{t'-r}(y-x') - p'_{t-r}(y-x))^2 e^{2R_1|y|} |u(r, y)|^{2\gamma} dy dr, \\ Q_{S,2,\delta,\eta_0}(s, t, x, t', x') &= \int_0^{(s-\delta)^+} \int 1(|y-x| \leq (t'-r)^{1/2-\eta_0} \vee 2|x-x'|) \\ &\quad \times (p'_{t'-r}(y-x') - p'_{t-r}(y-x))^2 e^{2R_1|y|} |u(r, y)|^{2\gamma} dy dr. \end{aligned}$$

Lemma 5.4 For all $K \in \mathbb{N}^{\geq K_1}$, $R > 2$ there is a $c_{5.4}(K, R)$ and an $N_{5.4} = N_2(K, \omega) \in \mathbb{N}$ a.s. so that for all $\eta_0, \eta_1 \in (1/R, 1/2)$, $\delta \in (0, 1]$, $\beta \in [0, 1/2]$ and $N, n \in \mathbb{N}$, for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, on

$$(5.10) \quad \{\omega : (t, x) \in Z(N, n, K, \beta), N \geq N_{5.4}\},$$

(5.11)

$$Q_{S,1,\delta,\eta_0}(s, t, x, t', x') \leq c_{5.4} 2^{4N_{5.4}(\omega)} \left[d^{2-\eta_1} + (d \wedge \sqrt{\delta})^{2-\eta_1} \delta^{-3/2} (d \wedge 1)^{4\gamma} \right] \text{ for all } s \leq t \leq t' \text{ and } x' \in \mathbb{R}.$$

Here $d = d((t', x'), (t, x))$.

Proof We let $N_{5.4}(K, \omega) = N_1(0, 3/4, K)$, that is we recall from Remark 5.3 that for $m = 0$, N_1 depends only on ξ and K and we take $\xi = 3/4$. We may assume $\delta < s$ as the left-hand side is 0 otherwise. Then for ω as in (5.10) and $s \leq t \leq t'$, Lemma 5.2 with $m = 0$ implies

$$\begin{aligned}
& Q_{S,1,\delta,\eta_0}(s, t, x, t', x') \\
& \leq C_{5.2}(\omega) \int_0^{(s-\delta)^+} \int 1(|y-x| > (t'-r)^{1/2-\eta_0} \vee 2|x-x'|)(p'_{t'-r}(y-x') - p'_{t-r}(y-x))^2 \\
& \quad \times e^{2R_1|y|} e^{2|y-x|} (2^{-N} \vee (\sqrt{t-r} + |y-x|))^{\gamma^{3/2}} dy ds \\
& \leq C_{5.2}(\omega) \int_0^{(s-\delta)^+} \int 1(|y-x| > (t'-r)^{1/2-\eta_0} \vee 2|x-x'|)(p'_{t'-r}(y-x') - p'_{t-r}(y-x))^2 \\
& \quad \times e^{2R_1K} e^{2(R_1+1)|y-x|} [2K^{\gamma^{3/4}} + 2|y-x|^{\gamma^{3/2}}] dy ds \\
& \leq C_{5.2}(\omega) c_0(K, R) \int_0^{s-\delta} (t-r)^{-3/2} \exp\left\{\frac{-\eta_1(t'-r)^{-2\eta_0}}{64}\right\} \left[1 \wedge \frac{d^2}{t-r}\right]^{1-\eta_1/2} dr.
\end{aligned}$$

In the last line we have used Lemma 4.4(b). Use the trivial bound (recall $r \leq s \leq t \leq t'$)

$$\exp\left\{\frac{-\eta_1(t'-r)^{-2\eta_0}}{64}\right\} \leq \exp\left\{\frac{-\eta_1(t'-t)^{-2\eta_0}}{128}\right\} + \exp\left\{\frac{-\eta_1(t-r)^{-2\eta_0}}{128}\right\},$$

and then Lemma 4.1 in the above, to bound $Q_{S,1,\delta,\eta_0}(s, t, x, t', x')$ by

$$\begin{aligned}
& C_{5.2}(\omega) c_0(K, R) \left[\int_0^{s-\delta} (t-r)^{-3/2} \left[1 \wedge \frac{d^2}{t-r}\right]^{1-\eta_1/2} dr \exp\left\{\frac{-\eta_1(t'-t)^{-2\eta_0}}{128}\right\} \right. \\
& \quad \left. + \int_0^{s-\delta} (t-r)^{-3/2} \left(1 \wedge \frac{d^2}{t-r}\right)^{1-\eta_1/2} \exp\left\{\frac{-\eta_1(t-r)^{-2\eta_0}}{128}\right\} dr \right] \\
& \leq C_{5.2}(\omega) c_1(K, R) \left[(d^2 \wedge \delta)^{1-\eta_1/2} \delta^{-3/2} \exp\left\{\frac{-\eta_1(t'-t)^{-2\eta_0}}{128}\right\} + C_1(R) \int_0^{s-\delta} \left(1 \wedge \frac{d^2}{t-r}\right)^{1-\eta_1/2} dr \right] \\
& \leq C_{5.2}(\omega) c_2(K, R) \left[(d \wedge \sqrt{\delta})^{2-\eta_1} \delta^{-3/2} (d \wedge 1)^{4\gamma} + d^{2-\eta_1} \right].
\end{aligned}$$

The final inequality with the sudden appearance of γ is crude but produces a form which will prove useful for this relative small contribution to the increments of F_δ . Now since we may set $\varepsilon_0 = 0$ in the formula for $C_{5.2}$ by Remark 5.3, the result follows. \blacksquare

Lemma 5.5 *Let $0 \leq m \leq \bar{m} + 1$ and assume (P_m) . For any $K \in \mathbb{N}^{\geq K_1}$, $R > 2$, $n \in \mathbb{N}$, $\varepsilon_0 \in (0, 1)$, and $\beta \in [0, 1/2]$ there is a $c_{5.5}(K, R)$ and $N_{5.5} = N_{5.5}(m, n, R, \varepsilon_0, K, \beta)(\omega) \in \mathbb{N}$ a.s. such that for any $\eta_1 \in (R^{-1}, 1/2)$, $\eta_0 \in (0, \eta_1/32)$, $\delta \in [a_n, 1]$, $N \in \mathbb{N}$, and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, on*

$$(5.12) \quad \{\omega : (t, x) \in Z(N, n, K, \beta), N \geq N_{5.5}\},$$

$$\begin{aligned}
& Q_{S,2,\delta,\eta_0}(s,t,x,t',x') \\
& \leq c_{5.5}(K,R)[a_n^{-2\varepsilon_0} + 2^{4N_{5.5}}] \left[d^{2-\eta_1} [\bar{\delta}_N^{(\gamma\gamma_m-3/2)\wedge 0} + a_n^{2\beta\gamma} \bar{\delta}_N^{\gamma-\frac{3}{2}}] \right. \\
& \quad \left. + (d \wedge \sqrt{\delta})^{2-\eta_1} \delta^{-3/2} [\bar{d}_N^{2\gamma\tilde{\gamma}_m} + a_n^{2\beta\gamma} \bar{d}_N^{2\gamma}] \right] \\
& \quad \text{for all } s \leq t \leq t' \leq K, |x'| \leq K+1.
\end{aligned}$$

Here $d = d((t,x), (t',x'))$, $\bar{d}_N = d \vee 2^{-N}$ and $\bar{\delta}_N = \delta \vee \bar{d}_N^2$. Moreover $N_{5.5}$ is stochastically bounded uniformly in (n,β) .

Proof. Let $\xi = 1 - (8R)^{-1} \in (15/16, 1)$ and define $N_{5.5} = N_1(m,n,\xi,\varepsilon_0,K,\beta)$ so that the last statement is immediate from (P_m) . We may assume $s \geq \delta$, or the left-hand side is 0. As $\delta \geq a_n$, when we use Lemma 5.2 to bound $u(r,y)$ in the integral defining $Q_{S,2,\delta,\eta_0}$, we have $d((r,y), (t,x)) \geq \sqrt{a_n}$ and so we may drop the max with $\sqrt{a_n}$. So for ω as in (5.12), $s \leq t \leq t'$ and $|x'| \leq K+1$, Lemma 5.2 implies that

$$\begin{aligned}
& Q_{S,2,\delta,\eta_0}(s,t,x,t',x') \\
& \leq C_{5.2} \int_0^{s-\delta} \int (p'_{t'-r}(y-x') - p'_{t-r}(y-x))^2 e^{2R_1K} e^{2(R_1+1)2(2K+1)y} dy \\
& \quad \times [2^{-N} \vee ((t-r)^{1/2} + (t'-r)^{1/2-\eta_0} \vee (2|x-x'|))]^{2\gamma\xi} \\
& \quad \times \left\{ [2^{-N} \vee ((t-r)^{1/2} + (t'-r)^{1/2-\eta_0} \vee (2|x-x'|))]^{\tilde{\gamma}_m-1} + a_n^\beta \right\}^{2\gamma} dr.
\end{aligned}$$

The harmless exponential factor comes from the $e^{|y-x|}$ factor in Lemma 5.2, the $E^{R_1|y|}$ in $Q_{S,2,\delta,\eta_0}$, the bound $|y| \leq |x| + |y-x|$ and the bounds on t' , $|x|$, $|x'|$ and $|y-x|$. Let $\gamma' = \gamma(1-2\eta_0)$. Recall that $t \leq t' \leq K$, $|x| \leq K$ and $|x'| \leq K+1$, so that $\sqrt{t-r} \leq K^{\eta_0}(t'-r)^{1/2-\eta_0}$ and $|x-x'| \leq (2K+1)|x-x'|^{1-2\eta_0}$. Use this and Lemma 4.4(a) to see that the above is at most

$$\begin{aligned}
& c_1(K)C_{5.2} \int_0^{s-\delta} (t-r)^{-3/2} \left(1 \wedge \frac{d^2}{t-r}\right) (2^{-2N\gamma} \vee (t-r)^{\gamma'} \vee |x-x'|^{2\gamma'})^\xi \\
(5.13) \quad & \quad \times \left[2^{-2N\gamma(\tilde{\gamma}_m-1)} \vee (t'-r)^{\gamma'(\tilde{\gamma}_m-1)} \vee |x'-x|^{2\gamma'(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma} \right] dr.
\end{aligned}$$

Note that

$$(5.14) \quad 2^{-2N\gamma} \vee (t'-r)^{\gamma'} \vee |x'-x|^{2\gamma'} \leq 2^{-2N\gamma'} \vee d^{2\gamma'} + (t-r)^{\gamma'} \leq 2[\bar{d}_N^{2\gamma'} \vee (t-r)^{\gamma'}].$$

Use this to bound the summands in (5.13) and conclude that

$$\begin{aligned}
& Q_{S,2,\delta,\eta_0}(s, t, x, t', x') \\
& \leq c_2(K)C_{5.2} \int_0^{s-\delta} (t-r)^{-3/2} \left(1 \wedge \frac{d^2}{t-r}\right) [\bar{d}_N^2 \vee (t-r)]^{\gamma'\xi} \\
& \quad \times \left[[\bar{d}_N^2 \vee (t-r)]^{\gamma'(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma} \right] dr \\
& \leq c_2(K)C_{5.2} \left\{ \int_0^{t-\bar{\delta}_N} (t-r)^{\gamma'\xi-3/2} \left(1 \wedge \frac{d^2}{t-r}\right) \left[(t-r)^{\gamma'(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma} \right] dr \right. \\
& \quad \left. + \int_0^{t-\delta} (t-r)^{-3/2} \left(1 \wedge \frac{d^2}{t-r}\right) dr \bar{d}_N^{2\gamma'\xi} \left[\bar{d}_N^{2\gamma'(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma} \right] \right\} \\
(5.15) \quad & \equiv c_2(K)C_{5.2} \{I_1 + I_2\}.
\end{aligned}$$

Apply Lemma 4.1(c) to see that

$$(5.16) \quad I_2 \leq (d \wedge \sqrt{\delta})^2 \delta^{-3/2} \bar{d}_N^{2\gamma'\xi} \left[\bar{d}_N^{2\gamma'(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma} \right].$$

In the integral defining I_1 we may drop the minimum with 1 and, adding a $\log(1/\bar{\delta}_N)$ factor just in case the exponent on u is -1 , we arrive at

$$\begin{aligned}
& I_1 \leq d^2 \left[\int_{\bar{\delta}_N}^t u^{\gamma'(\tilde{\gamma}_m+\xi-1)-5/2} du + a_n^{2\beta\gamma} \int_{\bar{\delta}_N}^t u^{\gamma'\xi-5/2} du \right] \\
(5.17) \quad & \leq c_3(K)d^2 \log(1/\bar{\delta}_N) \left[\bar{\delta}_N^{[\gamma'(\tilde{\gamma}_m+\xi-1)-3/2] \wedge 0} + a_n^{2\beta\gamma} \bar{\delta}_N^{\gamma'\xi-3/2} \right]
\end{aligned}$$

The $\log 1/\bar{\delta}_N$ is bounded by $c(R)d^{-\eta_1/2}$ for $\eta_1 > 1/R$. A bit of arithmetic shows that our conditions $\eta_0 \leq \eta_1/32$, $1-\xi = 1/8R$ and $\eta_1 > 1/R$ allow us to shift ξ to 1 and γ' to γ in the exponents on the right-hand sides of (5.16) and (5.17) at the cost of multiplying by $d^{-\eta_1/2}$. So using this, (5.16) and (5.17) in (5.15), we get

$$\begin{aligned}
& Q_{S,2,\delta,\eta_0}(s, t, x, t', x') \\
& \leq c_4(K, R)C_{5.2} \left\{ d^{2-\eta_1} \left[\bar{\delta}_N^{[\gamma\tilde{\gamma}_m-3/2] \wedge 0} + a_n^{2\beta\gamma} \bar{\delta}_N^{\gamma-3/2} \right] \right. \\
& \quad \left. + (d \wedge \sqrt{\delta})^{2-\eta_1} \delta^{-3/2} \left[\bar{d}_N^{2\gamma\tilde{\gamma}_m} + a_n^{2\beta\gamma} \bar{d}_N^{2\gamma} \right] \right\}.
\end{aligned}$$

The result follows from the definition of $C_{5.2}$ and the identity

$$(5.18) \quad [\gamma(\gamma_m \wedge 2) - 3/2] \wedge 0 = (\gamma\gamma_m - 3/2) \wedge 0$$

(use $\gamma > 3/4$ here). ■

Lemma 5.6 *Let $0 \leq m \leq \bar{m} + 1$ and assume (P_m) . For any $K \in \mathbb{N}^{\geq K_1}$, $R > 2$, $n \in \mathbb{N}$, $\varepsilon_0 \in (0, 1)$, and $\beta \in [0, 1/2]$ there is a $c_{5.6}(K)$ and $N_{5.6} = N_{5.6}(m, n, R, \varepsilon_0, K, \beta)(\omega) \in \mathbb{N}$ a.s. such that for any $\eta_1 \in (R^{-1}, 1/2)$, $\delta \in [a_n, 1]$, $N \in \mathbb{N}$, and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, on*

$$(5.19) \quad \{\omega : (t, x) \in Z(N, n, K, \beta), N \geq N_{5.6}\},$$

$$\begin{aligned}
& Q_{T,\delta}(s, s', t', x') \\
& \leq c_{5.6}(K)[a_n^{-2\varepsilon_0} + 2^{4N_{5.6}}]|s' - s|^{1-\frac{\eta_1}{2}} \left[\bar{\delta}_N^{(\gamma\gamma_m-3/2)\wedge 0} + a_n^{2\beta\gamma} \bar{\delta}_N^{\gamma-3/2} \right. \\
& \quad \left. + 1(\delta < \bar{d}_N^2) \delta^{-3/2} [\bar{d}_N^{2\gamma\tilde{\gamma}_m} + a_n^{2\beta\gamma} \bar{d}_N^{2\gamma}] \right] \\
(5.20) \quad & \text{for all } s \leq t \leq t', s' \leq t' \leq T_K, \text{ and } |x'| \leq K+1.
\end{aligned}$$

Here $d = d((t, x), (t', x'))$, $\bar{d}_N = d \vee 2^{-N}$ and $\bar{\delta}_N = \delta \vee \bar{d}_N^2$. Moreover $N_{5.6}$ is stochastically bounded uniformly in (n, β) .

Proof. Let $\xi = 1 - (2R)^{-1}$ and define $N_{5.6} = N_1(m, n, \xi, \varepsilon_0, K, \beta)$ so that the last statement is immediate from (P_m) . We may assume $s \vee s' \equiv \bar{s} \geq \delta$, or the left-hand side is 0. Let $\underline{s} = s \wedge s'$. We again use Lemma 5.2 to bound $|u(r, y)|$ in the integrand defining $Q_{T,\delta}$ and the maximum with $\sqrt{a_n}$ can be ignored as it is less than $\sqrt{t' - r}$ in the calculation below. So for ω as in (5.19) and s, t, s', t', x' as in (5.20), we have (note that $r \leq \bar{s} \leq t' \leq T_K$ so that Lemma 5.2 applies)

$$\begin{aligned}
& Q_{T,\delta}(s, s', t', x') \\
& \leq C_{5.2} \int_{(\underline{s}-\delta)^+}^{\bar{s}-\delta} \int p'_{t'-r}(y-x')^2 e^{2R_1 K} e^{2(R_1+1)|y-x|} [2^{-N} \vee (\sqrt{t'-r} + |y-x|)]^{2\gamma\xi} \\
& \quad \times \left[(2^{-N} \vee (\sqrt{t'-r} + |y-x|))^{\tilde{\gamma}_m-1} + a_n^\beta \right]^{2\gamma} dy dr.
\end{aligned}$$

Use Lemma 4.2, the inequality

$$(5.21) \quad 2^{-N} \vee (\sqrt{t'-r} + |y-x|) \leq \left(2^{-N} \vee |x-x'| \right) + \sqrt{t'-r} + |y-x'| \leq \bar{d}_N + \sqrt{t'-r} + |y-x'|,$$

and $e^{2(R_1+1)|y-x|} \leq c_0(K) e^{2(R_1+1)|y-x'|}$ to bound the above by

$$\begin{aligned}
& c_1(K) C_{5.2} \int_{(\underline{s}-\delta)^+}^{\bar{s}-\delta} \int (t'-r)^{-1} p_{2(t'-r)}(z)^2 e^{2(R_1+1)|z|} [\bar{d}_N^{2\gamma\xi} + (t'-r)^{\gamma\xi} + |z|^{2\gamma\xi}] \\
& \quad \times \left[(\bar{d}_N^{2\gamma(\tilde{\gamma}_m-1)} + (t'-r)^{\gamma(\tilde{\gamma}_m-1)} + |z|^{2\gamma(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma}) dz dr \right] \\
& \leq c_2(K) C_{5.2} \int_{(\underline{s}-\delta)^+}^{\bar{s}-\delta} (t'-r)^{-3/2} [\bar{d}_N^{2\gamma\xi} + (t'-r)^{\gamma\xi}] \left[(\bar{d}_N^{2\gamma(\tilde{\gamma}_m-1)} + (t'-r)^{\gamma(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma}) \right] dr \\
& \leq c_2(K) C_{5.2} \left\{ \int_{(\underline{s}-\delta)^+}^{\bar{s}-\delta} 1(r \leq t' - \bar{d}_N^2) \left[(t'-r)^{\gamma(\tilde{\gamma}_m+\xi-1)-3/2} + a_n^{2\beta\gamma} (t'-r)^{\gamma\xi-3/2} \right] dr \right. \\
& \quad \left. + \int_{(\underline{s}-\delta)^+}^{\bar{s}-\delta} 1(r > t' - \bar{d}_N^2) (t'-r)^{-3/2} dr \left[\bar{d}_N^{2\gamma(\tilde{\gamma}_m+\xi-1)} + a_n^{2\beta\gamma} \bar{d}_N^{2\gamma\xi} \right] \right\} \\
(5.22) \quad & \equiv c_2(K) C_{5.2} \{ J_1 + J_2 \}.
\end{aligned}$$

In the first line use

$$\int |z|^p p_u(z)^2 e^{\lambda|z|} dz \leq c(\lambda) u^{(p-1)/2} \quad \text{for } 0 \leq p \leq 2 \text{ and } \lambda > 0,$$

and recall dependence on R_1 is suppressed in our constants. Now

$$\begin{aligned} \int_{(\underline{s}-\delta)^+}^{\bar{s}-\delta} 1(r > t' - \bar{d}_N^2)(t' - r)^{-3/2} dr &\leq 1(\delta < \bar{d}_N^2)[(t' - \bar{s} + \delta)^{-3/2}|s' - s| \wedge 2(t' - \bar{s} + \delta)^{-1/2}] \\ &\leq 1(\delta < \bar{d}_N^2)2\delta^{-3/2}(|s' - s| \wedge \delta), \end{aligned}$$

and so

$$(5.23) \quad J_2 \leq 1(\delta < \bar{d}_N^2)2\delta^{-3/2}(|s' - s| \wedge \delta)\bar{d}_N^{-2\gamma(1-\xi)} \left[\bar{d}_N^{2\gamma\tilde{\gamma}_m} + a_n^{2\beta\gamma} \bar{d}_N^{2\gamma} \right]$$

$$(5.24) \quad \leq c_3(K)1(\delta < \bar{d}_N^2)\delta^{-3/2}(|s' - s| \wedge \delta)^{1-\frac{\eta_1}{2}} \left[\bar{d}_N^{2\gamma\tilde{\gamma}_m} + a_n^{2\beta\gamma} \bar{d}_N^{2\gamma} \right].$$

In the last line we used $\gamma(1-\xi) \leq 1-\xi \leq (2R)^{-1} < \eta_1/2$ and $|s' - s| \leq 2K$.

Turning to J_1 , let $p = \gamma(\tilde{\gamma}_m + \xi - 1) - 3/2$ or $\gamma\xi - 3/2$ for $0 \leq m-1 \leq \bar{m}$. Our bounds on γ and ξ (both are bigger than $3/4$) imply $p \in [\gamma\xi - 3/2, 1/2] \subset [-15/16, 1/2]$. If $p' = p \wedge 0$ and $0 \leq \varepsilon \leq -p'$, then

$$\begin{aligned} I(p) &\equiv \int_{(\underline{s}-\delta)^+}^{\bar{s}-\delta} 1(r \leq t' - \bar{d}_N^2)(t' - r)^p dr \\ &\leq \int_{(\underline{s}-\delta)^+}^{\bar{s}-\delta} 1(r \leq t' - \bar{d}_N^2)\sqrt{K}(t' - r)^{p'} dr \\ &\leq \sqrt{K} \min(|s' - s| \bar{\delta}_N^{p'}, \int_0^{|s'-s|} u^{p'} du) \\ &\leq 16\sqrt{K}|s' - s|^{p'+1} \min\left(\left(\frac{|s' - s|}{\bar{\delta}_N}\right)^{-p'}, 1\right) \quad (\text{use } p' \geq -15/16) \\ &\leq 16\sqrt{K}|s' - s|^{p'+1} \left(\frac{|s' - s|}{\bar{\delta}_N}\right)^{-p'-\varepsilon} \\ (5.25) \quad &= 16\sqrt{K}|s' - s|^{1-\varepsilon} \bar{\delta}_N^{\varepsilon+p'}. \end{aligned}$$

Define $q = p + \gamma(1-\xi)$, so that $q = \gamma\tilde{\gamma}_m - 3/2$ or $\gamma - 3/2$.

Case 1. $q \leq 0$.

Then $p' = p \leq 0$. If $\varepsilon = \gamma(1-\xi) \leq (2R)^{-1} < \eta_1/2$, then $\varepsilon + p' = q \leq 0$ and so (5.25) applies, and gives

$$(5.26) \quad I(p) \leq 16\sqrt{K}|s' - s|^{1-\varepsilon} \bar{\delta}_N^q \leq 16K|s' - s|^{1-\frac{\eta_1}{2}} \bar{\delta}_N^q.$$

Case 2. $q > 0$

Then $p' = (q - \gamma(1-\xi)) \wedge 0 \geq -\gamma(1-\xi)$. Let $\varepsilon = -p' \leq \gamma(1-\xi) \leq (2R)^{-1} < \eta_1/2$ in (5.25) and conclude

$$(5.27) \quad I(p) \leq 16\sqrt{K}|s' - s|^{1-\varepsilon} \leq 16K|s' - s|^{1-\frac{\eta_1}{2}}.$$

In either case we have shown that $I(p) \leq 16K|s' - s|^{1-\frac{\eta_1}{2}} \bar{\delta}_N^{q \wedge 0}$. This gives

$$(5.28) \quad J_1 \leq c_4(K)|s' - s|^{1-\frac{\eta_1}{2}} \left[\bar{\delta}_N^{(\gamma\tilde{\gamma}_m - 3/2) \wedge 0} + a_n^{2\beta\gamma} \bar{\delta}_N^{(\gamma - 3/2) \wedge 0} \right].$$

Put (5.24) and (5.28) into (5.22) and use (5.18) to complete the proof. ■

Notation. $d((s, t, x), (s', t', x')) = \sqrt{|s' - s|} + \sqrt{|t' - t|} + |x' - x|$.

Lemma 5.7 *Let c_0, c_1, c_2, k_0 be positive (universal constants), $\eta \in (0, 1/2)$, and $\Delta : \mathbb{N} \times (0, 1] \rightarrow \mathbb{R}_+$ satisfy $\Delta(n, 2^{-N+1}) \leq k_0 \Delta(n, 2^{-N})$ for all $n, N \in \mathbb{N}$. For $n \in \mathbb{N}$ and τ in a set S assume $\{Y_{\tau,n}(s, t, x) : (s, t, x) \in \mathbb{R}_+^2 \times \mathbb{R}\}$ is a real-valued continuous process. Assume for each (n, τ) , $K \in \mathbb{N}$, and $\beta \in [0, 1/2]$, there is an $N_0(\omega) = N_0(n, \eta, K, \tau, \beta)(\omega) \in \mathbb{N}$ a.s., stochastically bounded uniformly in (n, τ, β) , such that for any $N \in \mathbb{N}$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, and $s \leq K$, if $\tilde{d} = d((s, t, x), (s', t', x')) \leq 2^{-N}$, then*

$$(5.29) \quad P(|Y_{\tau,n}(s, t, x) - Y_{\tau,n}(s', t', x')| > \tilde{d}^{1-\eta} \Delta(n, 2^{-N}), (t, x) \in Z(N, n, K, \beta), N \geq N_0, t' \leq T_K) \leq c_0 \exp(-c_1 \tilde{d}^{-\eta c_2}).$$

Then there is an $N'_0 = N'_0(n, \eta, K, \tau, \beta) \in \mathbb{N}$ a.s., also stochastically bounded uniformly in (n, τ, β) , such that for all $N \geq N'_0(\omega)$, $(t, x) \in Z(N, n, K, \beta)(\omega)$, $\tilde{d} = d((s, t, x), (s', t', x')) \leq 2^{-N}$, $s \leq K$ and $t' \leq T_K$,

$$|Y_{\tau,n}(s, t, x) - Y_{\tau,n}(s', t', x')| \leq 2^7 k_0^3 \tilde{d}^{1-\eta} \Delta(n, 2^{-N}).$$

Proof. Let

$$M_{\ell, N} = M_{\ell, N}^{\tau, n, K, \beta} = \max\{|Y_{\tau,n}((i+e)2^{-2\ell}, (j+f)2^{-2\ell}, (k+g)2^{-\ell}) - Y_{\tau,n}(i2^{-2\ell}, j2^{-2\ell}, k2^{-\ell})| : (j2^{-2\ell}, k2^{-\ell}) \in Z(N, n, K+1, \beta), e, f = -4, -3, \dots, 4, i2^{-2\ell} \leq K+1, g = -2, -1, \dots, 2, (j+f)2^{-2\ell} \leq T_{K+1}, i, j, i+e, j+f \in \mathbb{Z}_+, k \in \mathbb{Z}\},$$

and

$$A_N = \{\omega : \exists \ell \geq N+3 \text{ s. t. } M_{\ell, N} \geq 2^{(3-\ell)(1-\eta)} \Delta(n, 2^{-N}), N \geq N_0(n, \eta, K+1, \tau, \beta)\}.$$

For i, j, k, e, f, g as in the definition of $M_{\ell, N}$ and $\ell \geq N+3$,

$$d(((i+e)2^{-2\ell}, (j+f)2^{-2\ell}, (k+g)2^{-\ell}), (i2^{-2\ell}, j2^{-2\ell}, k2^{-\ell})) \leq 2^{3-\ell} \leq 2^{-N}.$$

Therefore (5.29) implies that for some $c'_1 = c'_1(\eta) > 0$,

$$P(\cup_{N'=N}^{\infty} A_{N'}) \leq \sum_{N'=N}^{\infty} \sum_{\ell=N'+3}^{\infty} 5 \cdot 9^2 [2^{2\ell}(K+1) + 1]^2 [2^{\ell+1}(K+1) + 1] c_0 \exp(-c_1 (2^{3-\ell})^{-\eta c_2}) \leq c_3(K) \exp(-c'_1 2^{N\eta c_2}).$$

Let

$$N_2 = N_2(n, \eta, K, \tau, \beta) = \min\{N : \omega \in \cap_{N'=N}^{\infty} A_{N'}^c\}.$$

The above implies that

$$(5.30) \quad P(N_2 > N) = P(\cup_{N'=N}^{\infty} A_{N'}) \leq c_3(K) \exp(-c'_1 2^{N\eta c_2}).$$

Define

$$N'_0(n, \eta, K, \tau, \beta) = (N_0(n, \eta, K+1, \tau, \beta) \vee N_2(n, \eta, K, \tau, \beta)) + 3.$$

N'_0 is stochastically bounded uniformly in (n, τ, β) by (5.30) and the corresponding property of N_0 . Assume

$$(5.31) \quad N \geq N'_0, (t, x) \in Z(N, n, K, \beta), d((s, t, x), (s', t', x')) \leq 2^{-N}, s \leq K \text{ and } t' \leq T_K.$$

Define dyadic approximations by $s_\ell = \lfloor 2^{2\ell} s \rfloor 2^{-2\ell}$, $t_\ell = \lfloor 2^{2\ell} t \rfloor 2^{-2\ell}$, $x_\ell = \text{sgn}(x) \lfloor 2^\ell |x| \rfloor 2^{-\ell}$, and similarly define s'_ℓ , t'_ℓ and x'_ℓ for (s', t', x') . Choose (\hat{t}_0, \hat{x}_0) as in the definition of $(t, x) \in Z(N, n, K, \beta)$. Then $|x_\ell| \leq |x| \leq K$, $|x'_\ell| \leq |x'| \leq K + 1$, $s'_\ell \vee s_\ell \leq s' \vee s \leq K + 1$, $t'_\ell \vee t_\ell \leq t' \vee t \leq T_K$, and if $\ell \geq N$,

$$\begin{aligned} d((t'_\ell, x'_\ell), (\hat{t}_0, \hat{x}_0)) &\leq d((t'_\ell, x'_\ell), (t', x')) + d((t', x'), (t, x)) + d((t, x), (\hat{t}_0, \hat{x}_0)) \\ &\leq \sqrt{|t'_\ell - t'| + |x'_\ell - x'|} + 2^{1-N} \\ &\leq 2^{2-N}. \end{aligned}$$

This proves that

$$(5.32) \quad (t'_\ell, x'_\ell) \in Z(N-2, n, K+1, \beta) \subset Z(N-3, n, K+1, \beta) \text{ for all } \ell \geq N,$$

and even more simply one gets

$$(5.33) \quad (t_\ell, x_\ell) \in Z(N-3, n, K, \beta) \text{ for all } \ell \geq N.$$

In addition, the fact that $N \geq N'_0$ implies $\omega \in A_{N-3}^c$ and $N-3 \geq N_0$, which in turn implies

$$(5.34) \quad M_{\ell, N-3} \leq 2^{(3-\ell)(1-\eta)} \Delta(n, 2^{-(N-3)}) \text{ for all } \ell \geq N.$$

Choose $N' \geq N$ such that $2^{-N'-1} < d((s, t, x), (s', t', x')) \equiv \tilde{d} \leq 2^{-N'}$. Then $|x' - x| \leq 2^{-N'}$ which implies $x'_{N'} = x_{N'} + g2^{-N'}$ for $g \in \{-1, 0, 1\}$. Similarly $s'_{N'} = s_{N'} + e2^{-2N'}$ and $t'_{N'} = t_{N'} + f2^{-2N'}$ for $e, f \in \{-1, 0, 1\}$. In addition, $s_\ell = s_{\ell-1} + e4^{-\ell}$, $t_\ell = t_{\ell-1} + f4^{-\ell}$, and $x_\ell = x_{\ell-1} + g2^{-\ell}$ for some $e, f \in \{0, \dots, 3\}$ and $g \in \{-1, 0, 1\}$, and similarly for s'_ℓ , t'_ℓ and x'_ℓ . Let $w_\ell = (s_\ell, t_\ell, x_\ell)$ and $w'_\ell = (s'_\ell, t'_\ell, x'_\ell)$. Now use (5.32), (5.33), (5.34), the definition of $M_{\ell, N-3}$ and the continuity of $Y_{\tau, n}$ to see that for $(s, t, x), (s', t', x')$ as in (5.31),

$$\begin{aligned} &|Y_{\tau, n}(s, t, x) - Y_{\tau, n}(s', t', x')| \\ &\leq |Y_{\tau, n}(w'_{N'}) - Y_{\tau, n}(w_{N'})| + \sum_{\ell=N'+1}^{\infty} |Y_{\tau, n}(w'_\ell) - Y_{\tau, n}(w'_{\ell-1})| + |Y_{\tau, n}(w_\ell) - Y_{\tau, n}(w_{\ell-1})| \\ &\leq M_{N', N-3} + \sum_{\ell=N'+1}^{\infty} 2M_{\ell, N-3} \\ &\leq \left[2^{(3-N')(1-\eta)} + 2 \sum_{\ell=N'+1}^{\infty} 2^{(3-\ell)(1-\eta)} \right] \Delta(n, 2^{-(N-3)}) \quad (\text{by (5.34)}) \\ &\leq (36) 2^{-N'(1-\eta)} \Delta(n, 2^{-(N-3)}) \\ &\leq 2^7 k_0^3 \tilde{d}^{1-\eta} \Delta(n, 2^{-N}). \end{aligned}$$

■

Notation. Introduce

$$(5.35) \quad \bar{\Delta}_{u'_1}(m, n, \alpha, \varepsilon_0, 2^{-N}) = a_n^{-\varepsilon_0} \left[a_n^{-3\alpha/4} 2^{-N\gamma\tilde{\gamma}m} + (a_n^{\alpha/2} \vee 2^{-N})^{(\gamma_{m+1}-2)\wedge 0} + a_n^{-3\alpha/4+\beta\gamma} (a_n^{\alpha/2} \vee 2^{-N})^\gamma \right].$$

We often suppress the dependence on ε_0 and α .

Proposition 5.8 *Let $0 \leq m \leq \bar{m} + 1$ and assume (P_m) (in (5.5)). For any $n \in \mathbb{N}$, $\eta_1 \in (0, 1/2)$, $\varepsilon_0 \in (0, 1)$, $K \in \mathbb{N}^{\geq K_1}$, $\alpha \in [0, 1]$, and $\beta \in [0, 1/2]$, there is an $N_{5.8} = N_{5.8}(m, n, \eta_1, \varepsilon_0, K, \alpha, \beta)(\omega)$ in $\mathbb{N}^{\geq 2}$ a.s. such that for all $N \geq N_{5.8}$, $(t, x) \in Z(N, n, K, \beta)$, $t' \leq T_K$, $s \leq K$,*

$$d((s, t, x), (s', t', x')) \leq 2^{-N} \text{ implies that}$$

$$|F_{a_n^\alpha}(s, t, x) - F_{a_n^\alpha}(s', t', x')| \leq 2^{-86} d((s, t, x), (s', t', x'))^{1-\eta_1} \bar{\Delta}_{u'_1}(m, n, \alpha, \varepsilon_0, 2^{-N}).$$

Moreover $N_{5.8}$ is stochastically bounded, uniformly in (n, α, β) .

Proof. Let $R = 33\eta_1^{-1}$ and choose $\eta_0 \in (R^{-1}, \eta_1/32)$. Let $d = d((t, x), (t', x'))$, $\tilde{d} = d + \sqrt{|s' - s|}$, $\bar{d}_N = d \vee 2^{-N}$, $\bar{\delta}_{n,N} = a_n^\alpha \vee \bar{d}_N^2$ and (recall the Q 's are as in (5.9))

$$Q_{a_n^\alpha}(s, t, x, s', t', x') = Q_{T, a_n^\alpha}(s, s', t', x') + \sum_{i=1}^2 Q_{S, i, a_n^\alpha, \eta_0}(s, t, x, t', x').$$

By Lemmas 5.4, 5.5 and 5.6 with $\delta = a_n^\alpha$, there is a constant $c_1(K, \eta_1)$ and a random variable $N_2 = N_2(m, n, \eta_1, \varepsilon_0, K, \beta)(\omega)$, stochastically bounded uniformly in (n, β) , such that for all $N \in \mathbb{N}$ and (t, x) , on

$$(5.36) \quad \{\omega : (t, x) \in Z(N, n, K + 1, \beta), N \geq N_2\},$$

$$(5.37) \quad R_0^\gamma Q_{a_n^\alpha}(s, t, x, s', t', x')^{1/2} \leq c_1(K, \eta_1) [a_n^{-\varepsilon_0} + 2^{2N_2}] \bar{d}^{1-\eta_1/2} \left\{ a_n^{-3\alpha/4} [\bar{d}_N^{\gamma\tilde{\gamma}m} + a_n^{\beta\gamma} \bar{d}_N^\gamma] + \left(\sqrt{\bar{\delta}_{n,N}} \right)^{(\gamma\tilde{\gamma}m-3/2)\wedge 0} + a_n^{\beta\gamma} \left(\sqrt{\bar{\delta}_{n,N}} \right)^{\gamma-3/2} \right\}$$

for all $s \leq t \leq t'$, $s' \leq t' \leq T_K$, $|x'| \leq K + 2$.

Let $N_3 = (33/\eta_1)[N_2 + N_4(K, \eta_1)]$, where $N_4(K, \eta_1)$ is chosen large enough so that

$$(5.38) \quad c_1(K, \eta_1) [a_n^{-\varepsilon_0} + 2^{2N_2}] 2^{-\eta_1 N_3/4} \leq c_1(K, \eta_1) [a_n^{-\varepsilon_0} + 2^{2N_2}] 2^{-8N_2 - 8N_4} \leq a_n^{-\varepsilon_0} 2^{-104}.$$

Let

$$\Delta(m, n, \bar{d}_N) = 2^{-100} a_n^{-\varepsilon_0} \left\{ a_n^{-3\alpha/4} [\bar{d}_N^{\gamma\tilde{\gamma}m} + a_n^{\beta\gamma} \bar{d}_N^\gamma] + \left(\sqrt{\bar{\delta}_{n,N}} \right)^{(\gamma\tilde{\gamma}m-3/2)\wedge 0} + a_n^{\beta\gamma} \left(\sqrt{\bar{\delta}_{n,N}} \right)^{\gamma-3/2} \right\}.$$

Let $N' \in \mathbb{N}$ and assume $\tilde{d} \leq 2^{-N'}$. Use (5.37) and (5.38) to see that on

$$\{\omega : (t, x) \in Z(N, n, K + 1, \beta), N \geq N_3, N' \geq N_3\}$$

(which implies $|x'| \leq K + 2$),

$$(5.39) \quad R_0^\gamma Q_{a_n^\alpha}(s, t, x, s', t', x')^{1/2} \leq c_1(K, \eta_1) [a_n^{-\varepsilon_0} + 2^{2N_2}] 2^{-\eta_1 N'/4} \tilde{d}^{1-(3\eta_1/4)} 2^{100} a_n^{\varepsilon_0} \Delta(m, n, \bar{d}_N) \leq \tilde{d}^{1-(3\eta_1/4)} \frac{\Delta(m, n, \bar{d}_N)}{16} \quad \text{for all } s \leq t \leq t', s' \leq t' \leq T_K, |x'| \leq K + 2.$$

Combine this with (5.8), (3.2), the definition of $Q_{a_n^\alpha}$, and the Dubins-Schwarz theorem, to conclude that for $s \leq t \leq t'$, $s' \leq t'$, $d((s, t, x), (s', t', x')) \leq 2^{-N'}$,

$$\begin{aligned}
& P(|F_{a_n^\alpha}(s, t, x) - F_{a_n^\alpha}(s', t', x')| \geq d((s, t, x), (s', t', x'))^{1-\eta_1} \Delta(m, n, \bar{d}_N)/8, \\
& \quad (t, x) \in Z(N, n, K+1, \beta), N' \wedge N \geq N_3, t' \leq T_K) \\
& \leq 2P(\sup_{u \leq \bar{d}^{2-(3\eta_1/2)}(\Delta(m, n, \bar{d}_N)/16)^2} |B(u)| \geq \bar{d}^{1-\eta_1} \frac{\Delta(m, n, \bar{d}_N)}{16}) \\
& \leq 2P(\sup_{u \leq 1} |B(u)| \geq \bar{d}^{-\eta_1/4}) \\
(5.40) \quad & \leq c_0 \exp(-\bar{d}^{-\eta_1/2}/2).
\end{aligned}$$

Here $B(u)$ is a one-dimensional Brownian motion.

To handle $s > t$, recall that $F_{a_n^\alpha}(s, t, x) = F_{a_n^\alpha}(s, s \vee t, x)$. One easily checks that

$$\sqrt{|s \vee t - s' \vee t'|} \leq \sqrt{|s - s'|} + \sqrt{|t - t'|}$$

and hence $d((s, s \vee t, x), (s', s' \vee t', x')) \leq 2d((s, t, x), (s', t', x')) \equiv 2\tilde{d}$. So (5.40) implies that for $t \leq t'$ and all s, s', x' , if $d((s, t, x), (s', t', x')) \leq 2^{-N'-1}$, then

$$\begin{aligned}
& P(|F_{a_n^\alpha}(s, t, x) - F_{a_n^\alpha}(s', t', x')| \geq d((s, t, x), (s', t', x'))^{1-\eta_1} \Delta(m, n, \bar{d}_N)/4, \\
& \quad (t, x) \in Z(N, n, K+1, \beta), t' \leq T_K, N'+1 \geq N_3+1, N \geq N_3) \\
(5.41) \quad & \leq c_0 \exp(-\tilde{d}^{-\eta_1/2}/2).
\end{aligned}$$

If $(t, x) \in Z(N, n, K, \beta)$, $t' \leq t$ and $d = d((t, x), (t', x')) \leq 2^{-N}$, then we claim that $(t', x') \in Z(N-1, n, K+1, \beta)$. Indeed if (\hat{t}_0, \hat{x}_0) is as in the definition of $(t, x) \in Z(N, n, K, \beta)$, then $d((\hat{t}_0, \hat{x}_0), (t', x')) \leq 2^{-(N-1)}$. Also $|x'| \leq K+1$, $t' \leq t \leq T_K$, and the claim follows. Note also that as $d \leq 2^{-N}$, we have $\bar{d}_N = 2^{-N}$. An elementary argument using, $\gamma\tilde{\gamma}_k \leq 2$, shows that

$$(5.42) \quad \Delta(m, n, 2^{-N}) \geq 4^{-1} \Delta(m, n, 2^{-(N-1)}).$$

So, by interchanging (t', x') and (t, x) , and replacing N with $N-1$, (5.41) and (5.42) imply that for $t' \leq t$, $\tilde{d} \leq 2^{-N'}$ and $d \leq 2^{-N}$,

$$\begin{aligned}
& P(|F_{a_n^\alpha}(s, t, x) - F_{a_n^\alpha}(s', t', x')| \geq \bar{d}^{1-\eta_1} \Delta(m, n, 2^{-N}), \\
& \quad (t, x) \in Z(N, n, K, \beta), N' \wedge N \geq N_3+1) \\
& \leq P(|F_{a_n^\alpha}(s, t, x) - F_{a_n^\alpha}(s', t', x')| \geq \bar{d}^{1-\eta_1} \Delta(m, n, 2^{-(N-1)})/4, \\
& \quad (t', x') \in Z(N-1, n, K+1, \beta), N' \geq N_3+1, N-1 \geq N_3, t \leq T_K) \\
(5.43) \quad & \leq c_0 \exp(-\tilde{d}^{-\eta_1/2}/2).
\end{aligned}$$

If $N_5(m, n, \eta_1 \varepsilon_0, K, \beta)(\omega) = N_3(\omega) + 1$, then N_5 is stochastically bounded, uniformly in (n, β) . We have shown, (taking $N' = N$ in the above) that for all $(s, t, x), (s', t', x')$, if $\tilde{d} = d((s, t, x), (s', t', x')) \leq 2^{-N}$, then

$$\begin{aligned}
(5.44) \quad & P(|F_{a_n^\alpha}(s, t, x) - F_{a_n^\alpha}(s', t', x')| \geq \bar{d}^{1-\eta_1} \Delta(m, n, 2^{-N}), (t, x) \in Z(N, n, K, \beta), N \geq N_5, t' \leq T_K) \\
& \leq c_0 \exp(-\tilde{d}^{-\eta_1/2}/2).
\end{aligned}$$

Now apply Lemma 5.7 with $\tau = \alpha \in [0, 1]$, $Y_{\tau, n} = F_{a_n^\alpha}$ and $k_0 = 2^2$, the latter by (5.42). (5.44) shows that (5.29) holds with $N_0 = N_5$. (The implicit restriction $K \geq K_1$ in (5.44) from Lemmas 5.4-5.6 is illusory as increasing K only strengthens (5.44).) Therefore there is an $N_{5.8}(\omega) = N_{5.8}(m, n, \eta_1, \varepsilon_0, K, \alpha, \beta) \geq 2$, stochastically bounded uniformly in (n, α, β) , such that for $N \geq N_{5.8}$, $(t, x) \in Z(N, n, K, \beta)$, if $t' \leq T_K$, $s \leq K$ and $\tilde{d} = d((s, t, x), (s', t', x')) \leq 2^{-N}$, then

$$(5.45) \quad |F_{a_n^\alpha}(s, t, x) - F_{a_n^\alpha}(s', t', x')| \leq 2^{13} \Delta(m, n, 2^{-N}) \tilde{d}^{1-m}.$$

Note that

$$(5.46) \quad \begin{aligned} \Delta(m, n, 2^{-N}) &= 2^{-100} a_n^{-\varepsilon_0} \left[a_n^{-3\alpha/4} 2^{-N\gamma\tilde{\gamma}m} + (a_n^{\alpha/2} \vee 2^{-N})^{(\gamma\tilde{\gamma}m - 3/2) \wedge 0} \right. \\ &\quad \left. + a_n^{\beta\gamma} (a_n^{-3\alpha/4} 2^{-N\gamma} + (a_n^{\alpha/2} \vee 2^{-N})^{\gamma-3/2}) \right] \\ &\leq 2^{-99} a_n^{-\varepsilon_0} \left[a_n^{-3\alpha/4} 2^{-N\gamma\tilde{\gamma}m} + (a_n^{\alpha/2} \vee 2^{-N})^{(\gamma_{m+1} - 2) \wedge 0} \right. \\ &\quad \left. + a_n^{\beta\gamma - \frac{3\alpha}{4}} (2^{-N} \vee a_n^{\alpha/2})^\gamma \right]. \end{aligned}$$

Use this in (5.45) to complete the proof. ■

Since $F_\delta(t, t, x) = -u'_{1,\delta}(t, x)$ (see Remark 4.2), the following Corollary is immediate.

Corollary 5.9 *Let $0 \leq m \leq \bar{m} + 1$ and assume (P_m) (in (5.5)). Let $n, \eta_1, \varepsilon_0, K, \alpha$ and β be as in Proposition 5.8. For all $N \geq N_{5.8}$, $(t, x) \in Z(N, n, K, \beta)$ and $t' \leq T_K$,*

$$\begin{aligned} d((t, x), (t', x')) \leq 2^{-N} \text{ implies that} \\ |u'_{1, a_n^\alpha}(t, x) - u'_{1, a_n^\alpha}(t', x')| \leq 2^{-85} d((t, x), (t', x'))^{1-m} \bar{\Delta}_{u'_1}(m, n, \alpha, \varepsilon_0, 2^{-N}). \end{aligned}$$

We will need to modify the bound in Lemma 5.6 to control $|u'_{1,\delta} - u'_{1,a_n}|$. Note that if $\delta \geq a_n$ and $s = t - \delta + a_n$ then

$$(5.47) \quad \begin{aligned} u'_{1,\delta}(t, x) &= \frac{d}{dx} P_\delta(u_{(t-\delta)^+})(x) = \frac{d}{dx} P_{t-s+a_n}(u_{(s-a_n)^+})(x) \\ &= -F_{a_n}(s, t, x) \\ &= -F_{a_n}(t - \delta + a_n, t, x). \end{aligned}$$

Therefore the key will be a bound on $|F_{a_n}(s, t, x) - F_{a_n}(t, t, x)|$ in which the hypothesis (for Proposition 5.8) $\sqrt{t-s} \leq 2^{-N}$ is weakened substantially.

Lemma 5.10 *Let $0 \leq m \leq \bar{m} + 1$ and assume (P_m) . For any $K \in \mathbb{N}^{\geq K_1}$, $R > 2$, $n \in \mathbb{N}$, $\varepsilon_0 \in (0, 1)$, and $\beta \in [0, 1/2]$ there is a $c_{5.10}(K)$ and $N_{5.10} = N_{5.10}(m, n, R, \varepsilon_0, K, \beta)(\omega) \in \mathbb{N}$ a.s. such that for any $\eta_1 \in (R^{-1}, 1/2)$, $N \in \mathbb{N}$, and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, on*

$$(5.48) \quad \{\omega : (t, x) \in Z(N, n, K, \beta), N \geq N_{5.10}\},$$

$$\begin{aligned}
& Q_{T,a_n}(s, t, t, x) \\
& \leq c_{5.10}(K)[a_n^{-2\varepsilon_0} + 2^{4N_{5.10}}] \left\{ |t-s|^{1-\eta_1/4} [((t-s) \vee a_n)^{\gamma\tilde{\gamma}_m-3/2} + a_n^{2\beta\gamma} ((t-s) \vee a_n)^{\gamma-3/2}] \right. \\
& \quad \left. + 1(a_n < 2^{-2N})((t-s) \wedge a_n) a_n^{-3/2} 2^{N\eta_1/2} [2^{-2N\gamma\tilde{\gamma}_m} + a_n^{2\beta\gamma} 2^{-2N\gamma}] \right\}
\end{aligned}$$

for all $s \leq t$.

Moreover $N_{5.10}$ is stochastically bounded uniformly in (n, β) .

Proof. Let $\xi = 1 - (4\gamma R)^{-1}$ and define $N_{5.10} = N_1(m, n, \xi(R), \varepsilon_0, K, \beta)$ so that the last statement is immediate from (P_m) . We may assume $t \geq a_n$. Argue as in the derivation of (5.22) with $x' = x$, $s' = t' = t$ and $\delta = a_n$. The only difference is that instead of the upper bound in (5.21), $x' = x$ allows to use the trivial upper bound $2^{-N} = \sqrt{t-r} + |y-x|$, that is \bar{d}_N is replaced by 2^{-N} . So (5.22) now implies that if

$$J_1 = \int_{(s-a_n)^+}^{t-a_n} 1(r < t - 2^{-2N}) [(t-r)^{\gamma(\tilde{\gamma}_m+\xi-1)-3/2} + a_n^{2\beta\gamma} (t-r)^{\gamma\xi-3/2}] dr,$$

and

$$J_2 = \int_{(s-a_n)^+}^{t-a_n} 1(r \geq t - 2^{-2N}) (t-r)^{-3/2} dr 2^{-2N\gamma\xi} [2^{-2N\gamma(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma}],$$

then

$$(5.49) \quad Q_{T,a_n}(s, t, t, x) \leq c(K)C_{5.2}[J_1 + J_2].$$

As in the derivation of (5.23), now with $\bar{d}_N = 2^{-N}$, $\delta = a_n$ and $s' = t$, we get

$$(5.50) \quad J_2 \leq 1(a_n < 2^{-2N}) 2(a_n \wedge (t-s)) a_n^{-3/2} 2^{2N\gamma(1-\xi)} [2^{-2N\gamma\tilde{\gamma}_m} + a_n^{2\beta\gamma} 2^{-2N\gamma}].$$

For J_1 , let $p = \gamma(\tilde{\gamma}_m+\xi-1)-3/2$ or $p = \gamma\xi-3/2$. Our choice of ξ and R implies $p \in [-15/16, 1/2]$ and so, considering $p \geq 0$ and $p < 0$ separately, we arrive at

$$\begin{aligned}
\int_{(s-a_n)^+}^{t-a_n} (t-r)^p dr & \leq 16[(t-s+a_n)^{p+1} - a_n^{p+1}] \leq 16(2^{p\vee 0})(t-s)((t-s) \vee a_n)^p \\
& \leq 24(t-s)((t-s) \vee a_n)^p.
\end{aligned}$$

Therefore

$$\begin{aligned}
(5.51) \quad J_1 & \leq 24(t-s) \left[((t-s) \vee a_n)^{\gamma(\tilde{\gamma}_m+\xi-1)-3/2} + a_n^{2\beta\gamma} ((t-s) \vee a_n)^{\gamma\xi-3/2} \right] \\
& \leq 24(t-s)^{1-\gamma(1-\xi)} \left[((t-s) \vee a_n)^{\gamma\tilde{\gamma}_m-3/2} + a_n^{2\beta\gamma} ((t-s) \vee a_n)^{\gamma-3/2} \right].
\end{aligned}$$

Put (5.50) and (5.51) into (5.49), noting that $\gamma(1-\xi(R)) = (4R)^{-1} < \eta_1/4$ and $t-s \leq K$, to complete the proof. \blacksquare

Proposition 5.11 *Let $0 \leq m \leq \bar{m} + 1$ and assume (P_m) . For any $n \in \mathbb{N}$, $\eta_1 \in (0, 1/2)$, $\varepsilon_0 \in (0, 1)$, $K \in \mathbb{N}^{\geq K_1}$, and $\beta \in [0, 1/2]$, there is an $N_{5.11} = N_{5.11}(m, n, \eta_1, \varepsilon_0, K, \beta)(\omega) \in \mathbb{N}$ a.s. such that for all $N \geq N_{5.11}$, $(t, x) \in Z(N, n, K, \beta)$, $s \leq t$ and $\sqrt{t-s} \leq N^{-4/\eta_1}$ implies that*

$$\begin{aligned} & |F_{a_n}(s, t, x) - F_{a_n}(t, t, x)| \\ & \leq 2^{-81} a_n^{-\varepsilon_0} \left\{ 2^{-N(1-\eta_1)} (a_n^{1/2} \vee 2^{-N})^{\gamma_{m+1-2} \wedge 0} \right. \\ & \quad + 2^{N\eta_1} a_n^{-1/4} \left(\frac{2^{-N}}{\sqrt{a_n}} + 1 \right) \left(2^{-N\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma} (\sqrt{a_n} \vee 2^{-N})^\gamma \right) \\ & \quad \left. + (t-s)^{(1-\eta_1)/2} \left((\sqrt{t-s} \vee \sqrt{a_n})^{\gamma\tilde{\gamma}_m - \frac{3}{2}} + a_n^{\beta\gamma} (\sqrt{t-s} \vee \sqrt{a_n})^{\gamma - \frac{3}{2}} \right) \right\}. \end{aligned}$$

Moreover $N_{5.11}$ is stochastically bounded, uniformly in (n, β) .

Proof. Apply Lemma 5.10 with $R = 2/\eta_1$ so that on

$$\{\omega : (t, x) \in Z(N, n, K, \beta), N \geq N_{5.10}(m, n, 2/\eta_1, \varepsilon_0, K, \beta)\},$$

for $s \leq t$,

$$\begin{aligned} & R_0^\gamma Q_{T, a_n}(s, t, t, x)^{1/2} \\ (5.52) \quad & \leq c_1(K) R_0^\gamma [a_n^{-\varepsilon_0} + 2^{2N_{5.10}}] \left\{ (\sqrt{t-s})^{\eta_1/4} (\sqrt{t-s})^{1-\frac{\eta_1}{2}} [(\sqrt{t-s} \vee \sqrt{a_n})^{\gamma\tilde{\gamma}_m - \frac{3}{2}} \right. \\ & \quad \left. + a_n^{\beta\gamma} (\sqrt{t-s} \vee \sqrt{a_n})^{\gamma - \frac{3}{2}}] \right. \\ & \quad \left. + 2^{-N\eta_1/4} a_n^{-1/4} 2^{N\eta_1/2} [2^{-N\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma} 2^{-N\gamma}] \right\}. \end{aligned}$$

Let $N_2(m, n, \eta_1, \varepsilon_0, K, \beta)(\omega) = \frac{8}{\eta_1} [N_{5.10} + N_0(K)]$, where $N_0(K) \in \mathbb{N}$ is chosen large enough so that

$$\begin{aligned} (5.53) \quad & c_1(K) R_0^\gamma [a_n^{-\varepsilon_0} + 2^{2N_{5.10}}] 2^{-\frac{\eta_1}{4} N_2} \leq c_1(K) R_0^\gamma [a_n^{-\varepsilon_0} + 2^{2N_{5.10}}] 2^{-2N_{5.10} - 2N_0(K)} \\ & \leq 2^{-100} a_n^{-\varepsilon_0}. \end{aligned}$$

It follows from (5.52) and (5.53) that for $N \geq N_2$, $(t, x) \in Z(N, n, K, \beta)$, $s \leq t$, and $\sqrt{t-s} \leq 2^{-N_2}$,

$$\begin{aligned} & R_0^\gamma Q_{T, a_n}(s, t, t, x)^{1/2} \\ & \leq 2^{-100} a_n^{-\varepsilon_0} \left\{ (\sqrt{t-s})^{1-\frac{\eta_1}{2}} [(\sqrt{t-s} \vee \sqrt{a_n})^{\gamma\tilde{\gamma}_m - \frac{3}{2}} + a_n^{\beta\gamma} (\sqrt{t-s} \vee \sqrt{a_n})^{\gamma - \frac{3}{2}}] \right. \\ & \quad \left. + a_n^{-1/4} 2^{N\eta_1/2} [2^{-N\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma} 2^{-N\gamma}] \right\} \\ & \equiv (\sqrt{t-s})^{1-\frac{\eta_1}{2}} \Delta_1(m, n, \sqrt{t-s} \vee \sqrt{a_n}) + 2^{N\eta_1/2} \Delta_2(m, n, 2^{-N}), \end{aligned}$$

that is Δ_1 and Δ_2 are defined as the appropriate two terms from the previous line. Combine this with (3.2), (5.8) (now with $t' = t = s'$, $x = x'$, so the second integral there is 0) and the

Dubins-Schwarz theorem to see that if $B(\cdot)$ is a standard 1-dimensional Brownian motion, then

$$\begin{aligned}
(5.54) \quad & P(|F_{a_n}(s, t, x) - F_{a_n}(t, t, x)| \geq (\sqrt{t-s})^{1-\eta_1} \Delta_1(m, n, \sqrt{t-s} \vee \sqrt{a_n}) + 2^{N\eta_1} \Delta_2(m, n, 2^{-N}), \\
& \quad \quad \quad (t, x) \in Z(N, n, K, \beta), N \geq N_2, \sqrt{t-s} \leq 2^{-N_2}) \\
& \leq P(\sup_{u \leq 1} |B(u)| \geq (\sqrt{t-s})^{-\eta_1/2} \wedge 2^{N\eta_1/2}) 1(t-s \leq 1) \\
& \leq c_0 \exp\left\{-\frac{1}{2}[(t-s)^{-\eta_1/2} \wedge 2^{N\eta_1}]\right\}.
\end{aligned}$$

Let $\ell_N = 2^{2(N+3)} N^{-8/\eta_1}$ and set

$$\begin{aligned}
M_N(\omega) = \max \left\{ \frac{|F_{a_n}(i2^{-2(N+2)}, j2^{-2(N+2)}, k2^{-(N+2)}) - F_{a_n}(j2^{-2(N+2)}, j2^{-2(N+2)}, k2^{-(N+2)})|}{(\sqrt{j-i}2^{-(N+2)})^{1-\eta_1} \Delta_1(m, n, (\sqrt{j-i}2^{-(N+2)}) \vee \sqrt{a_n}) + 2^{N\eta_1} \Delta_2(m, n, 2^{-N})} : \right. \\
\left. 0 \leq j-i \leq \ell_N, (j2^{-2(N+2)}, k2^{-(N+2)}) \in Z(N, n, K, \beta), i, j \in \mathbb{Z}_+, k \in \mathbb{Z} \right\}.
\end{aligned}$$

If $N_3 = 2^{N_2}$, then

$$(5.55) \quad N \geq N_3 \Rightarrow N^{-4/\eta_1} \leq 2^{-N_2-1} \Rightarrow \sqrt{\ell_N} 2^{-N-2} = 2N^{-4/\eta_1} \leq 2^{-N_2}.$$

The fact that $M_N = 0$ if $\ell_N < 1$, (5.54), and (5.55) imply

$$\begin{aligned}
& P(M_N \geq 1, N \geq N_3) \\
& \leq (K+1)^2 2^{4(N+2)} (2K+1) 2^{N+2} c_0 \exp\left\{-\frac{1}{2}((\ell_N 2^{-2(N+2)})^{-\eta_1/2} \wedge 2^{N\eta_1})\right\} 1(\ell_N \geq 1) \\
& \leq c_1 K^3 2^{5N} \exp\left\{-\frac{1}{2}((\sqrt{\ell_N} 2^{-N}) \vee 2^{-N})^{-\eta_1}\right\} 1(\ell_N \geq 1) \\
& \leq c_1 K^3 2^{5N} \exp\left\{-2^{-5/2} N^4\right\} \quad (\text{recall } \eta_1 < 1/2).
\end{aligned}$$

If $A_N = \{M_N \geq 1, N \geq N_3\}$ and

$$N_4 = N_4(m, n, \eta_1, \varepsilon_0, K, \beta)(\omega) = \min\{N : \omega \in \cap_{N'=N}^{\infty} A_{N'}^c\},$$

then

$$\begin{aligned}
(5.56) \quad & P(N_4 > N) = P(\cup_{N'=N}^{\infty} A_{N'}) \leq c_1 K^3 \sum_{N'=N}^{\infty} 2^{5N'} \exp\left\{-2^{-5/2} (N')^4\right\} \\
& \leq c_2(K) \exp(-N^4/6).
\end{aligned}$$

Let $N_5(\eta_1)$ be large enough so that

$$(5.57) \quad N \geq N_5 \Rightarrow 2^{1-N} \leq N^{-4/\eta_1}.$$

Define

$$N_{5.11}(m, n, \eta_1, \varepsilon_0, K, \beta) = (N_{5.8} \vee N_3 \vee N_4 \vee N_5) + 2.$$

It follows from Proposition 5.8, (5.56), and the definition of N_3 that $N_{5.11}$ is stochastically bounded uniformly in (n, β) . Assume

$$(5.58) \quad N \geq N_{5.11}, (t, x) \in Z(N, n, K, \beta), s \leq t \text{ and } \sqrt{t-s} \leq N^{-4/\eta_1}.$$

Case 1. $\sqrt{t-s} \geq 2^{1-N}$.

The condition $N \geq N_{5.11}$ implies $\omega \in A_{N-2}^c$ and $N-2 \geq N_3$, which in turn implies

$$(5.59) \quad M_{N-2} < 1.$$

Let $s_\ell = \lfloor 2^{2\ell} s \rfloor 2^{-2\ell}$, $t_\ell = \lfloor 2^{2\ell} t \rfloor 2^{-2\ell}$ and $x_\ell = \text{sgn}(x) \lfloor 2^\ell |x| \rfloor 2^{-\ell}$, be the usual dyadic approximations to s, t and x , respectively, and let (\hat{t}_0, \hat{x}_0) be as in the definition of $(t, x) \in Z(N, n, K, \beta)$. Then

$$d((t_N, x_N), (\hat{t}_0, \hat{x}_0)) \leq 2^{-N} + \sqrt{t-t_N} + |x-x_N| \leq 2^{2-N}, \quad t_N \leq t \leq T_K, \quad |x_N| \leq |x| \leq K,$$

and so

$$(5.60) \quad (t_N, x_N) \in Z(N-2, n, K, \beta).$$

Write

$$(5.61) \quad \begin{aligned} |F_{a_n}(s, t, x) - F_{a_n}(t, t, x)| &\leq \left[|F_{a_n}(s, t, x) - F_{a_n}(s_N, t_N, x_N)| + |F_{a_n}(t, t, x) - F_{a_n}(t_N, t_N, x_N)| \right] \\ &\quad + \left[|F_{a_n}(s_N, t_N, x_N) - F_{a_n}(t_N, t_N, x_N)| \right] \\ &\equiv T_1 + T_2. \end{aligned}$$

The fact that $(t, x) \in Z(N, n, K, \beta)$, $t_N \leq t \leq T_K$, $s \leq t \leq K$,

$$d((t, t, x), (t_N, t_N, x_N)) \vee d((s, t, x), (s_N, t_N, x_N)) \leq 3(2^{-N}) \leq 2^{-(N-2)},$$

and $N-2 \geq N_{5.8}$, allows us to use Proposition 5.8 and infer that

$$(5.62) \quad T_1 \leq 2^{-85} 2^{-(N-2)(1-\eta)} \bar{\Delta}_{u'_1}(m, n, 1, \varepsilon_0, 2^{-(N-2)}).$$

For T_2 we have from $N \geq N_5$, (5.57), and the last part of (5.58),

$$\sqrt{t_N - s_N} \leq \sqrt{t-s} + 2^{1-N} \leq 2N^{-4/\eta} \leq \sqrt{\ell_{N-2}} 2^{-N}.$$

In view of (5.60) and (5.59), this implies

$$(5.63) \quad \begin{aligned} T_2 &\leq M_{N-2} \left[\sqrt{t_N - s_N}^{1-\eta} \Delta_1(m, n, \sqrt{t_N - s_N} \vee \sqrt{a_n}) + 2^{(N-2)\eta} \Delta_2(m, n, 2^{-(N-2)}) \right] \\ &\leq \left[\sqrt{t_N - s_N}^{1-\eta} \Delta_1(m, n, \sqrt{t_N - s_N} \vee \sqrt{a_n}) + 2^{(N-2)\eta} \Delta_2(m, n, 2^{-(N-2)}) \right]. \end{aligned}$$

As $t-s \geq 2^{2-2N}$ (recall this defines Case 1), we have

$$t-s \leq (t-t_N) + (t_N-s_N) \leq 2^{-2N} + (t_N-s_N) \leq \frac{1}{4}(t-s) + (t_N-s_N),$$

and so

$$(5.64) \quad t_N - s_N \geq \frac{1}{2}(t-s).$$

More simply,

$$(5.65) \quad t_N - s_N \leq t-s + 2^{1-2N} \leq 2(t-s).$$

Use (5.64) and (5.65) in (5.63) and then combine the result with (5.62) and (5.61) to conclude that

$$\begin{aligned}
& |F_{a_n}(s, t, x) - F_{a_n}(t, t, x)| \\
& \leq 2^{-85} 2^{-(N-2)(1-\eta_1)} a_n^{-\varepsilon_0} \left[(a_n^{1/2} \vee 2^{-(N-2)})^{(\gamma_{m+1}-2) \wedge 0} \right. \\
& \quad \left. + a_n^{-3/4} \left[2^{-(N-2)\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma} (a_n^{1/2} \vee 2^{-(N-2)})^\gamma \right] \right] \\
(5.66) \quad & + 2^{-99} a_n^{-\varepsilon_0} (\sqrt{t-s})^{1-\eta_1} \left[(\sqrt{t-s} \vee \sqrt{a_n})^{\gamma\tilde{\gamma}_m - \frac{3}{2}} + a_n^{\beta\gamma} (\sqrt{t-s} \vee \sqrt{a_n})^{\gamma - \frac{3}{2}} \right] \\
& + 2^{-100} a_n^{-\varepsilon_0 - \frac{1}{4}} 2^{(N-2)\eta_1} \left[2^{-(N-2)\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma} 2^{-(N-2)\gamma} \right].
\end{aligned}$$

Next use

$$2^{-N(1-\eta_1)} a_n^{-3/4} + 2^{N\eta_1} a_n^{-1/4} = a_n^{-1/4} 2^{N\eta_1} \left[\frac{2^{-N}}{\sqrt{a_n}} + 1 \right]$$

to combine the first and third terms in (5.66) and conclude, after a bit of arithmetic, that

$$\begin{aligned}
& |F_{a_n}(s, t, x) - F_{a_n}(t, t, x)| \\
& \leq 2^{-81} a_n^{-\varepsilon_0} \left\{ \left[a_n^{-1/4} 2^{N\eta_1} \left(\frac{2^{-N}}{\sqrt{a_n}} + 1 \right) \left(2^{-N\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma} (a_n^{1/2} \vee 2^{-N})^\gamma \right) \right. \right. \\
& \quad \left. \left. + 2^{-N(1-\eta_1)} (a_n^{1/2} \vee 2^{-N})^{(\gamma_{m+1}-2) \wedge 0} \right] \right. \\
(5.67) \quad & \left. + (\sqrt{t-s})^{1-\eta_1} \left((\sqrt{t-s} \vee \sqrt{a_n})^{\gamma\tilde{\gamma}_m - \frac{3}{2}} + a_n^{\beta\gamma} (\sqrt{t-s} \vee \sqrt{a_n})^{\gamma - \frac{3}{2}} \right) \right\}.
\end{aligned}$$

Case 2. $\sqrt{t-s} \leq 2^{1-N}$.

As $(t, x) \in Z(N-1, n, K, \beta)$ (by (5.58)), $s \leq t \leq K$, $N-1 \geq N_{5,8}$, and $d((s, t, x), (t, t, x)) \leq 2^{-(N-1)}$, we may use Proposition 5.8 with $\alpha = 1$ to conclude

$$\begin{aligned}
& |F_{a_n}(s, t, x) - F_{a_n}(t, t, x)| \\
& \leq 2^{-86} (\sqrt{t-s})^{1-\eta_1} \bar{\Delta}_{u_1'}(m, n, 1, \varepsilon_0, 2^{-(N-1)}) \\
& \leq 2^{-83} 2^{-N(1-\eta_1)} a_n^{-\varepsilon_0} \left[a_n^{-3/4} 2^{-N\gamma\tilde{\gamma}_m} + (a_n^{1/2} \vee 2^{-N})^{(\gamma_{m+1}-2) \wedge 0} + a_n^{-3/4+\beta\gamma} (a_n^{1/2} \vee 2^{-N})^\gamma \right] \\
& \leq 2^{-83} a_n^{-\varepsilon_0} \left\{ a_n^{-1/4} 2^{N\eta_1} \left(\frac{2^{-N}}{\sqrt{a_n}} \right) \left(2^{-N\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma} (a_n^{1/2} \vee 2^{-N})^\gamma \right) + 2^{-N(1-\eta_1)} (a_n^{1/2} \vee 2^{-N})^{(\gamma_{m+1}-2) \wedge 0} \right\}
\end{aligned}$$

which is bounded by the first term on the right-hand side of (5.67). \blacksquare

We also need an analogue of Proposition 5.8 for $G_{a_n^\alpha}$. A subset of the arguments in Lemma 3.1 shows that

$$(5.68) \quad G_\delta(s, t, x) = \int_0^{(s-\delta)^+} \int p_{(t \vee s)-r}(y-x) D(r, y) W(dr, dy) \text{ for all } s \text{ a.s. for all } (t, x),$$

which is just the analogue of the expression for F_δ , (3.6), with p_{t-r} in place of p'_{t-r} . Although we only will need bounds on $G_{a_n^\alpha}(s, t, x) - G_{a_n^\alpha}(t, t, x)$ (and for $\sqrt{t-s}$ small as in Proposition 5.8), this seems to require bounds on the analogues of the three types of square functions handled in Lemmas 5.4, 5.5 and 5.6, but now with no derivatives on the Gaussian densities. This results in some simplification and a smaller singularity in a_n^α . We omit the proof of the following result as the details are quite similar to those used to establish Proposition 5.8.

Proposition 5.12 *Let $0 \leq m \leq \bar{m} + 1$ and assume (P_m) (in (5.5)). For any $n \in \mathbb{N}$, $\eta_1 \in (0, 1/2)$, $\varepsilon_0 \in (0, 1)$, $K \in \mathbb{N}^{\geq K_1}$, $\alpha \in [0, 1]$, and $\beta \in [0, 1/2]$, there is an $N_{5.12} = N_{5.12}(m, n, \eta_1, \varepsilon_0, K, \alpha, \beta)$ in \mathbb{N} a.s. such that for all $N \geq N_{5.12}$, $(t, x) \in Z(N, n, K, \beta)$, $s \leq t$ and $\sqrt{t-s} \leq 2^{-N}$,*

$$|G_{a_n^\alpha}(s, t, x) - G_{a_n^\alpha}(t, t, x)| \leq 2^{-92}(t-s)^{\frac{1}{2}(1-\eta_1)} a_n^{-\varepsilon_0} a_n^{-\alpha/4} \\ \times \left[(a_n^{\alpha/2} \vee 2^{-N})^{\gamma \tilde{\gamma}_m} + a_n^{\beta \gamma} (a_n^{\alpha/2} \vee 2^{-N})^\gamma \right].$$

We need to use our global modulus of continuity for u'_{1, a_n^α} (Corollary 5.9) to get a modulus for u_{1, a_n^α} itself. This is of course easy for spatial increments, but a key observation is that it is possible to also use control of the spatial derivatives to get a better modulus on the temporal increments.

Notation. Define

$$(5.69) \quad \bar{\Delta}_{u_1}(m, n, \alpha, \varepsilon_0, 2^{-N}) = a_n^{-\varepsilon_0 - 3\alpha/4} \left[a_n^\beta a_n^{3\alpha/4} + a_n^{\beta \gamma} (a_n^{\alpha/2} \vee 2^{-N})^{\gamma+1} \right. \\ \left. + (a_n^{\alpha/2} \vee 2^{-N})^{\gamma \tilde{\gamma}_m + 1} + 1(m \geq \bar{m}) a_n^{3\alpha/4} (a_n^{\alpha/2} \vee 2^{-N}) \right].$$

Dependence on α or ε_0 is often suppressed.

If $\eta > 0$ let $N'_{5.13}(\eta)$ be the smallest natural number such that $2^{1-N} \leq N^{-\frac{4}{\eta}}$ whenever $N \geq N'_{5.13}(\eta)$.

Proposition 5.13 *Let $0 \leq m \leq \bar{m} + 1$ and assume (P_m) . For any $n \in \mathbb{N}$, $\eta_1 \in (0, 1/2)$, $\varepsilon_0, \varepsilon_1 \in (0, 1)$, $K \in \mathbb{N}^{\geq K_1}$, $\alpha \in [0, 1]$ and $\beta \in [0, 1/2]$, there is an $N_{5.13} = N_{5.13}(m, n, \eta_1, \varepsilon_0, K, \alpha, \beta) \in \mathbb{N}$ a.s. so that for all $N \geq N_{5.13}$, n, α satisfying*

$$(5.70) \quad a_n \leq 2^{-2(N_{5.11}(m, n, \eta_1/2, \varepsilon_0, K, \beta) + 1)} \wedge 2^{-2(N'_{5.13}(\eta_1 \varepsilon_1) + 1)}, \quad \text{and } \alpha \geq \varepsilon_1,$$

$(t, x) \in Z(N, n, K, \beta)$, $t' \leq T_K$, if $d((t, x), (t', x')) \leq 2^{-N}$, then

$$|u_{1, a_n^\alpha}(t, x) - u_{1, a_n^\alpha}(t', x')| \leq 2^{-90} d((t, x), (t', x'))^{1-\eta_1} \bar{\Delta}_{u_1}(m, n, \alpha, \varepsilon_0, 2^{-N}).$$

Moreover $N_{5.13}$ is stochastically bounded uniformly in $n \in \mathbb{N}$, $\alpha \in [0, 1]$ and $\beta \in [0, 1/2]$.

Remarks. Although n appears on both sides of (5.70), the stochastic boundedness of $N_{5.11}$ ensures it will hold for infinitely many n .

The non-increasing behavior of $N'_{5.13}(\eta_1 \varepsilon_1)$ in ε_1 shows (5.70) becomes stronger as the lower bound on α , ε_1 , goes to 0. This effectively rules out $\alpha = 0$ from the above conclusion. This is not surprising as allowing α to approach 0 allows, a_n^α to move away from the value a_n where the definition of $Z(N, n, K, \beta)$ ensures some control on u'_{1, a_n} .

Proof. Let

$$N''_{5.13}(m, n, \eta_1, \varepsilon_0, K, \alpha, \beta) = ((2N_{5.8})(m, n, \eta_1/2, \varepsilon_0, K+1, \alpha, \beta) \vee N_{5.12}(m, n, \eta_1, \varepsilon_0, K+1, \alpha, \beta)) + 1.$$

Clearly $N''_{5.13}$ is stochastically bounded uniformly in (n, α, β) . Assume (5.70) and

$$(5.71) \quad N \geq N''_{5.13}, \quad (t, x) \in Z(N, n, K, \beta), \quad t' \leq T_K \quad \text{and} \quad d((t, x), (t', x')) \leq 2^{-N}.$$

As in the proof of Proposition 5.8, $(t', x') \in Z(N-1, n, K+1, \beta)$, and by interchanging (t, x) with (t', x') , N with $N-1$ and K with $K+1$, in the argument below (again as in the proof of Proposition 5.8) we may assume without loss of generality that $t' \leq t$. Indeed, this is the reason for having $K+1$ and adding 1 in our definition of $N''_{5.13}$.

Recall that

$$(5.72) \quad G_{a_n^\alpha}(t', t, x) = P_{t-t'+a_n^\alpha}(u((t' - a_n^\alpha, \cdot)))(x) = P_{t-t'}(u_{1, a_n^\alpha}(t', \cdot))(x),$$

and so

$$(5.73) \quad \begin{aligned} |u_{1, a_n^\alpha}(t', x') - u_{1, a_n^\alpha}(t, x)| &\leq |u_{1, a_n^\alpha}(t', x') - u_{1, a_n^\alpha}(t', x)| + |u_{1, a_n^\alpha}(t', x) - P_{t-t'}(u_{1, a_n^\alpha}(t', \cdot))(x)| \\ &\quad + |G_{a_n^\alpha}(t', t, x) - G_{a_n^\alpha}(t, t, x)| \\ &\equiv T_1 + T_2 + T_3. \end{aligned}$$

For T_1 , let (\hat{t}_0, \hat{x}_0) be as in the definition of $(t, x) \in Z(N, n, K, \beta)$. For y between x and x' , $d((t', y), (t, x)) \leq 2^{-N}$, and also $d((\hat{t}_0, \hat{x}_0), (t, x)) \leq 2^{-N}$. Therefore by Corollary 5.9 (twice) with $\eta_1/2$ in place of η_1 ,

$$(5.74) \quad \begin{aligned} |u'_{1, a_n^\alpha}(t', y)| &\leq |u'_{1, a_n^\alpha}(t', y) - u'_{1, a_n^\alpha}(t, x)| + |u'_{1, a_n^\alpha}(t, x) - u'_{1, a_n^\alpha}(\hat{t}_0, \hat{x}_0)| \\ &\quad + |u'_{1, a_n^\alpha}(\hat{t}_0, \hat{x}_0) - u'_{1, a_n}(\hat{t}_0, \hat{x}_0)| + a_n^\beta \\ &\leq 2^{-84} 2^{-N(1-\frac{\eta_1}{2})} \bar{\Delta}_{u'_1}(m, n, \alpha, \varepsilon_0, 2^{-N}) \\ &\quad + |F_{a_n}(\hat{t}_0 - a_n^\alpha + a_n, \hat{t}_0, \hat{x}_0) - F_{a_n}(\hat{t}_0, \hat{t}_0, \hat{x}_0)| + a_n^\beta. \end{aligned}$$

We have used (5.47) in the last line.

We now use Proposition 5.11 to control the F increment in (5.74). Choose N' so that

$$(5.75) \quad 2^{-N'-1} \leq \sqrt{a_n} \leq 2^{-N'}.$$

(5.70) implies $\sqrt{a_n} \leq 2^{-N_{5.11}(m, n, \frac{\eta_1}{2}, \varepsilon_0, K, \beta)-1}$ and so

$$(5.76) \quad N' \geq N_{5.11}(m, n, \eta_1/2, \varepsilon_0, K, \beta).$$

In addition, (5.70) implies $2^{-N'-1} \leq \sqrt{a_n} \leq 2^{-N'_{5.13}(\eta_1 \varepsilon_1)-1}$ and so $N' \geq N'_{5.13}(\eta_1 \varepsilon_1)$ which in turn implies

$$(5.77) \quad a_n^{\alpha/2} \leq 2^{-N'\alpha} \leq 2^{-N'\varepsilon_1} \leq N'^{-\frac{4\varepsilon_1}{\eta_1 \varepsilon_1}} = N'^{-\frac{4}{\eta_1}}.$$

Since

$$|u(\hat{t}_0, \hat{x}_0)| \leq a_n = a_n \wedge (\sqrt{a_n} 2^{-N'}) \quad (\text{by (5.75)}),$$

we see that $(\hat{t}_0, \hat{x}_0) \in Z(N', n, K, \beta)$. (5.76) and (5.77) allow us to apply Proposition 5.11 with N' in place of N , (\hat{t}_0, \hat{x}_0) in place of (t, x) , $\eta_1/2$ in place of η_1 , and $s = \hat{t}_0 - a_n^\alpha + a_n$, and deduce

$$\begin{aligned} &|F_{a_n}(\hat{t}_0 - a_n^\alpha + a_n, \hat{t}_0, \hat{x}_0) - F_{a_n}(\hat{t}_0, \hat{t}_0, \hat{x}_0)| \\ &\leq 2^{-78} a_n^{-\varepsilon_0} \left[\sqrt{a_n}^{(1-\frac{\eta_1}{2})} \sqrt{a_n}^{(\gamma_{m+1}-2)\wedge 0} + \sqrt{a_n}^{-\frac{\eta_1+1}{2}} \left(\sqrt{a_n}^{\gamma \tilde{\gamma}^m} + a_n^{\beta \gamma} \sqrt{a_n}^\gamma \right) \right. \\ &\quad \left. + a_n^{\frac{\alpha}{2}(1-\frac{\eta_1}{2})} \left(a_n^{\frac{\alpha}{2}(\gamma \tilde{\gamma}^m - \frac{3}{2})} + a_n^{\beta \gamma} a_n^{\frac{\alpha}{2}(\gamma - \frac{3}{2})} \right) \right]. \end{aligned}$$

The middle term in the square brackets is bounded by the last term because $\sqrt{a_n} \leq a_n^{\frac{\alpha}{2}}$. Therefore

$$(5.78) \quad \begin{aligned} & |F_{a_n}(\hat{t}_0 - a_n^\alpha + a_n, \hat{t}_0, \hat{x}_0) - F_{a_n}(\hat{t}_0, \hat{t}_0, \hat{x}_0)| \\ & \leq 2^{-77} a_n^{-\varepsilon_0} \left[\sqrt{a_n}^{(1-\frac{\eta_1}{2})} \sqrt{a_n}^{(\gamma_{m+1}-2)\wedge 0} + a_n^{\frac{\alpha}{2}(1-\frac{\eta_1}{2})} a_n^{-\frac{3\alpha}{4}} \left(a_n^{\frac{\alpha}{2}\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma} a_n^{\frac{\alpha}{2}\gamma} \right) \right]. \end{aligned}$$

We also have

$$\sqrt{a_n}^{1-\frac{\eta_1}{2}+(\gamma_{m+1}-2)\wedge 0} \leq (a_n^{\frac{\alpha}{2}} \vee 2^{-N})^{1-\frac{\eta_1}{2}+(\gamma_{m+1}-2)\wedge 0},$$

because the above exponent is positive since $\eta_1 < 1/2$ and $\gamma > 3/4$. Use this bound in (5.78) and then insert the result into (5.74) to conclude that for any y between x and x' ,

$$(5.79) \quad \begin{aligned} & |u'_{1,a_n^\alpha}(t', y)| \\ & \leq 2^{-77} a_n^{-\varepsilon_0} \left\{ 2^{-N(1-\frac{\eta_1}{2})} (a_n^{\frac{\alpha}{2}} \vee 2^{-N})^{(\gamma_{m+1}-2)\wedge 0} \right. \\ & \quad + 2^{-N(1-\frac{\eta_1}{2})} a_n^{-\frac{3\alpha}{4}} \left[2^{-N\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma} (a_n^{\frac{\alpha}{2}} \vee 2^{-N})^\gamma \right] \\ & \quad + (a_n^{\frac{\alpha}{2}} \vee 2^{-N})^{1-\frac{\eta_1}{2}+(\gamma_{m+1}-2)\wedge 0} \\ & \quad \left. + a_n^{\frac{\alpha}{2}(1-\frac{\eta_1}{2})} a_n^{-\frac{3\alpha}{4}} \left(a_n^{\frac{\alpha}{2}\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma} a_n^{\frac{\alpha}{2}\gamma} \right) \right\} + a_n^\beta \\ & \leq 2^{-76} a_n^{-\varepsilon_0} (a_n^{\frac{\alpha}{2}} \vee 2^{-N})^{1-\frac{\eta_1}{2}} \left[(a_n^{\frac{\alpha}{2}} \vee 2^{-N})^{(\gamma_{m+1}-2)\wedge 0} + a_n^{-\frac{3\alpha}{4}} (a_n^{\frac{\alpha}{2}} \vee 2^{-N})^{\gamma\tilde{\gamma}_m} \right. \\ & \quad \left. + a_n^{-\frac{3\alpha}{4}} a_n^{\beta\gamma} (a_n^{\frac{\alpha}{2}} \vee 2^{-N})^\gamma \right] + a_n^\beta \\ & \equiv 2^{-76} \tilde{\Delta}_{u_1}(m, n, \alpha, \varepsilon_0, \eta_1, a_n^{\frac{\alpha}{2}} \vee 2^{-N}) + a_n^\beta. \end{aligned}$$

Note that $\tilde{\Delta}_{u_1}$ is monotone increasing in the $2^{-N} \vee a_n^{\frac{\alpha}{2}}$ variable due to the positivity of the exponents (since $\eta_1 < 1/2$). The Mean Value Theorem now shows that

$$(5.80) \quad T_1 \leq \left[a_n^\beta + 2^{-76} \tilde{\Delta}_{u_1}(m, n, \alpha, \varepsilon_0, \eta_1, a_n^{\frac{\alpha}{2}} \vee 2^{-N}) \right] |x - x'|.$$

Recalling that $t' \leq t$ and that (from (5.71)) $N \geq N_{5.12}$ and $\sqrt{t-t'} \leq 2^{-N}$, we may apply Proposition 5.12 and infer

$$(5.81) \quad T_3 \leq 2^{-92} (t-t')^{\frac{1}{2}(1-\eta_1)} a_n^{-\varepsilon_0-\alpha/4} \left[(a_n^{\alpha/2} \vee 2^{-N})^{\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma} (a_n^{\alpha/2} \vee 2^{-N})^\gamma \right].$$

For T_2 , let $\{B(s) : s \geq 0\}$ be a one-dimensional Brownian motion, starting at x under P_x . Assume first that

$$(5.82) \quad |B(t-t') - x| \leq 2^{-\frac{3}{2}N_{5.8}}.$$

Recalling that $N_{5.8} \geq 2$ and (5.71) (which implies $N \geq 2N_{5.8}$), we have

$$\begin{aligned} d((t', B(t-t')), (t, x)) & \leq \sqrt{t-t'} + 2^{-\frac{3}{2}N_{5.8}} \leq 2^{-N} + 2^{-\frac{3}{2}N_{5.8}} \\ & \leq 2^{-2N_{5.8}} + 2^{-\frac{3}{2}N_{5.8}} \\ & \leq 2^{-N_{5.8}}. \end{aligned}$$

Define a random $N' \in \{N_{5.8}, \dots, N\}$ by

- (i) if $d((t', B(t-t')), (t, x)) \leq 2^{-N}$ then $N' = N$;
- (ii) if $d((t', B(t-t')), (t, x)) > 2^{-N}$ then $2^{-N'-1} < d((t', B(t-t')), (t, x)) \leq 2^{-N'}$.

In case (ii) we have $2^{-N'-1} \leq 2^{-N} + |B(t-t') - x|$, and so

$$(5.83) \quad 2^{-N'} \leq 2^{1-N} + 2|B(t-t') - x|,$$

a result which is trivial in case (i). If y is between x and $B(t-t')$ we may argue as in (5.79), but now using $(t, x) \in Z(N', n, K, \beta)$, to see that

$$(5.84) \quad |u'_{1, a_n^\alpha}(t', y)| \leq 2^{-76} \tilde{\Delta}_{u_1}(m, n, \alpha, \varepsilon_0, \eta_1, a_n^{\frac{\alpha}{2}} \vee 2^{-N'}) + a_n^\beta.$$

Use (5.83) and the monotonicity of $\tilde{\Delta}_{u_1}$ observed above to see that

$$(5.85) \quad \begin{aligned} & a_n^{\varepsilon_0} \tilde{\Delta}_{u_1}(m, n, \alpha, \varepsilon_0, \eta_1, a_n^{\frac{\alpha}{2}} \vee 2^{-N'}) \\ & \leq 8 \left[a_n^{\frac{\alpha}{2}} + 2^{-N} + |B(t-t') - x| \right]^{1-\frac{\eta_1}{2}} \\ & \quad \times \left\{ (a_n^{\frac{\alpha}{2}} + 2^{-N} + |B(t-t') - x|)^{(\gamma_{m+1}-2) \wedge 0} + a_n^{-\frac{3\alpha}{4}} \left[(a_n^{\frac{\alpha}{2}} + 2^{-N} + |B(t-t') - x|)^{\gamma \tilde{\gamma}_m} \right. \right. \\ & \quad \left. \left. + a_n^{\beta\gamma} (a_n^{\frac{\alpha}{2}} + 2^{-N} + |B(t-t') - x|)^\gamma \right] \right\}. \end{aligned}$$

Use (5.85) in (5.84) and then the Mean Value Theorem to obtain (the expectation is over B alone— $N_{5.8}$ remains fixed—and we are dropping a number of small constants)

$$(5.86) \quad \begin{aligned} & E_x(1(|B(t-t') - x| \leq 2^{-\frac{3}{2}N_{5.8}}) |u_{1, a_n^\alpha}(t', B(t-t')) - u_{1, a_n^\alpha}(t', x)|) \\ & \leq E_0(|B(t-t')| \left\{ a_n^\beta + a_n^{-\varepsilon_0} [a_n^{\frac{\alpha}{2}(1-\frac{\eta_1}{2})} + 2^{-N(1-\frac{\eta_1}{2})} + |B(t-t')|^{1-\frac{\eta_1}{2}}] \left[a_n^{-3\alpha/4} [a_n^{\frac{\alpha}{2}} + 2^{-N} + |B(t-t')|]^{\gamma \tilde{\gamma}_m} \right. \right. \\ & \quad \left. \left. + (a_n^{\alpha/2} \vee 2^{-N})^{(\gamma_{m+1}-2) \wedge 0} + a_n^{-3\alpha/4} a_n^{\beta\gamma} (a_n^{\alpha/2} + 2^{-N} + |B(t-t')|)^\gamma \right] \right\} \\ & \leq c_1 \sqrt{t-t'} \left\{ a_n^\beta + a_n^{-\varepsilon_0} [(a_n^{\frac{\alpha}{2}} \vee 2^{-N})^{(1-\frac{\eta_1}{2})} + (t-t')^{\frac{1}{2}(1-\frac{\eta_1}{2})}] \left[a_n^{-3\alpha/4} (a_n^{\frac{\alpha}{2}} \vee 2^{-N} + \sqrt{t-t'})^{\gamma \tilde{\gamma}_m} \right. \right. \\ & \quad \left. \left. + (a_n^{\alpha/2} \vee 2^{-N})^{(\gamma_{m+1}-2) \wedge 0} + a_n^{-3\alpha/4} a_n^{\beta\gamma} (a_n^{\alpha/2} \vee 2^{-N} + \sqrt{t-t'})^\gamma \right] \right\} \\ & \leq c_2 \sqrt{t-t'} \left\{ a_n^\beta + a_n^{-\varepsilon_0} (a_n^{\frac{\alpha}{2}} \vee 2^{-N})^{(1-\frac{\eta_1}{2})} \left[(a_n^{\alpha/2} \vee 2^{-N})^{(\gamma_{m+1}-2) \wedge 0} + a_n^{-3\alpha/4} (a_n^{\alpha/2} \vee 2^{-N})^{\gamma \tilde{\gamma}_m} \right. \right. \\ & \quad \left. \left. + a_n^{-3\alpha/4} a_n^{\beta\gamma} (a_n^{\alpha/2} \vee 2^{-N})^\gamma \right] \right\} \quad (\text{since } \sqrt{t-t'} \leq 2^{-N}) \end{aligned}$$

$$= c_2 \sqrt{t-t'} [a_n^\beta + \tilde{\Delta}_{u_1}(m, n, \alpha, \varepsilon_0, \eta_1, a_n^{\frac{\alpha}{2}} \vee 2^{-N})].$$

To handle the complementary set to that on the left-hand side of (5.86), note that for $K \geq K_1$ and $t' \leq T_K$,

$$|u_{1, a_n^\alpha}(t', y)| \leq E_y(|u((t' - a_n^\alpha)^+, B(a_n^\alpha))|) \leq 2KE_y(e^{|B(a_n^\alpha)|}) \leq 2Ke^{1+|y|}.$$

This and the fact that $\sqrt{t-t'} \leq 2^{-2N_{5.8}}$ imply that

$$\begin{aligned}
& E_x(1(|B(t-t') - x| > 2^{-\frac{3}{2}N_{5.8}})|u_{1,a_n^\alpha}(t', B(t-t')) - u_{1,a_n^\alpha}(t', x)|) \\
& \leq P_0(|B(t-t')| > 2^{-\frac{3}{2}N_{5.8}})^{1/2} 8K e E_x(e^{2|B(t-t')|} + e^{2|x|})^{1/2} \\
& \leq c_3(K)P_0(|B(1)| > (t-t')^{-1/8})^{1/2} \quad (\text{since } |x| \leq K \text{ by (5.71)}) \\
& \leq c_4(K)(t-t') \\
(5.87) \quad & \leq c_5(K)\sqrt{t-t'}\tilde{\Delta}_{u_1}(m, n, \alpha, \varepsilon_0, \eta_1, a_n^{\frac{\alpha}{2}} \vee 2^{-N}),
\end{aligned}$$

where in the last line we use

$$\begin{aligned}
& \tilde{\Delta}_{u_1}(m, n, \alpha, \varepsilon_0, \eta_1, a_n^{\frac{\alpha}{2}} \vee 2^{-N}) \geq (a_n^{\frac{\alpha}{2}} \vee 2^{-N})^{1-\frac{\eta_1}{2}} \geq 2^{-N} \geq \sqrt{t-t'}. \\
(5.86) \text{ and (5.87) imply}
\end{aligned}$$

$$(5.88) \quad T_2 \leq c_6(K)\sqrt{t-t'}[a_n^\beta + \tilde{\Delta}_{u_1}(m, n, \alpha, \varepsilon_0, \eta_1, a_n^{\frac{\alpha}{2}} \vee 2^{-N})].$$

Use (5.80), (5.81) and (5.88) in (5.73) to conclude

$$\begin{aligned}
& |u_{1,a_n^\alpha}(t', x') - u_{1,a_n^\alpha}(t, x)| \\
& \leq c_7(K)d((t, x), (t', x'))[a_n^\beta + \tilde{\Delta}_{u_1}(m, n, \alpha, \varepsilon_0, \eta_1, a_n^{\frac{\alpha}{2}} \vee 2^{-N})] \\
& \quad + 2^{-92}(t'-t)^{\frac{1}{2}(1-\eta_1)}a_n^{-\varepsilon_0}a_n^{\alpha/2}a_n^{-3\alpha/4} \left[(a_n^{\alpha/2} \vee 2^{-N})^{\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma}(a_n^{\alpha/2} \vee 2^{-N})^\gamma \right].
\end{aligned}$$

since $d((t, x), (t', x')) \leq 2^{-N}$, a bit of arithmetic shows the above is at most

$$\begin{aligned}
(5.89) \quad & (c_7(K)2^{-N\frac{\eta_1}{2}} + 2^{-92})d((t, x), (t', x'))^{1-\eta_1}a_n^{-\varepsilon_0-3\alpha/4} \\
& \times \left\{ (a_n^{\alpha/2} \vee 2^{-N}) \left[(a_n^{\alpha/2} \vee 2^{-N})^{\gamma\tilde{\gamma}_m} + a_n^{\beta\gamma}(a_n^{\alpha/2} \vee 2^{-N})^\gamma \right. \right. \\
& \quad \left. \left. + a_n^{3\alpha/4}(a_n^{\alpha/2} \vee 2^{-N})^{(\gamma_{m+1}-2)\wedge 0} \right] + a_n^{3\alpha/4}a_n^\beta \right\}.
\end{aligned}$$

Choose $N_1(K, \eta_1)$ so that

$$2^{-N_1\eta_1/2}c_7(K) \leq 2^{-92},$$

and define $N_{5.13} = N_{5.13}'' \vee N_1$, which is clearly stochastically bounded uniformly in $(n, \alpha, \beta) \in \mathbb{N} \times [0, 1] \times [0, \frac{1}{2}]$. Assume $N \geq N_{5.13}$. Note that if $m < \bar{m}$, then

$$\begin{aligned}
(5.90) \quad & a_n^{3\alpha/4}(a_n^{\alpha/2} \vee 2^{-N})^{(\gamma_{m+1}-2)\wedge 0} = a_n^{3\alpha/4}(a_n^{\alpha/2} \vee 2^{-N})^{\gamma\gamma_m - \frac{3}{2}} \\
& \leq (a_n^{\alpha/2} \vee 2^{-N})^{\gamma\gamma_m} \\
& \leq (a_n^{\alpha/2} \vee 2^{-N})^{\gamma\tilde{\gamma}_m},
\end{aligned}$$

and so the last term in square brackets in (5.89) is bounded by the first term in the same brackets.

If $m \geq \bar{m}$, the left-hand side of (5.90) is $a_n^{3\alpha/4}$. Therefore if $N \geq N_{5.13}$ we conclude that

$$\begin{aligned}
& |u_{1,a_n^\alpha}(t', x') - u_{1,a_n^\alpha}(t, x)| \\
& \leq 2^{-90}d((t, x), (t', x'))^{1-\eta_1}a_n^{-\varepsilon_0-3\alpha/4} \\
& \quad \times \left[a_n^{\beta+3\alpha/4} + a_n^{\beta\gamma}(a_n^{\alpha/2} \vee 2^{-N})^{\gamma+1} + (a_n^{\alpha/2} \vee 2^{-N})^{\gamma\tilde{\gamma}_m+1} + 1(m \geq \bar{m})(a_n^{\alpha/2} \vee 2^{-N})a_n^{3\alpha/4} \right] \\
& = 2^{-90}d((t, x), (t', x'))^{1-\eta_1}\bar{\Delta}_{u_1}(m, n, \alpha, \varepsilon_0, 2^{-N}).
\end{aligned}$$

■

We also require an analogue of the bound on increments on u_{1,a_n^α} (Proposition 5.13) for u_{2,a_n^α} .

Notation.

$$\begin{aligned}\bar{\Delta}_{1,u_2}(m, n, \varepsilon_0, 2^{-N}) &= a_n^{-\varepsilon_0} 2^{-N\gamma} \left[(a_n^{1/2} \vee 2^{-N})^{\gamma(\tilde{\gamma}_m - 1)} + a_n^{\beta\gamma} \right] \\ \bar{\Delta}_{2,u_2}(m, n, \alpha, \varepsilon_0) &= a_n^{-\varepsilon_0} \left[a_n^{\frac{\alpha}{2}(\gamma\tilde{\gamma}_m - \frac{1}{2})} + a_n^{\beta\gamma} a_n^{\frac{\alpha}{2}(\gamma - \frac{1}{2})} \right].\end{aligned}$$

We often will suppress the dependence on ε_0 and α below.

Proposition 5.14 *Let $0 \leq m \leq \bar{m} + 1$ and assume (P_m) . For any $n \in \mathbb{N}$, $\eta_1 \in (0, 1/2)$, $\varepsilon_0 \in (0, 1)$, $K \in \mathbb{N}^{\geq K_1}$, $\alpha \in [0, 1]$, and $\beta \in [0, 1/2]$, there is an $N_{5.14} = N_{5.14}(m, n, \eta_1, \varepsilon_0, K, \alpha, \beta)(\omega) \in \mathbb{N}$ a.s. such that for all $N \geq N_{5.14}$, $(t, x) \in Z(N, n, K, \beta)$, and $t' \leq T_K$,*

$$\begin{aligned}d \equiv d((t, x), (t', x')) &\leq 2^{-N} \text{ implies that} \\ |u_{2,a_n^\alpha}(t, x) - u_{2,a_n^\alpha}(t', x')| &\leq 2^{-89} \left[d^{\frac{1-\eta_1}{2}} \bar{\Delta}_{1,u_2}(m, n, \varepsilon_0, 2^{-N}) + d^{1-\eta_1} \bar{\Delta}_{2,u_2}(m, n, \alpha, \varepsilon_0) \right].\end{aligned}$$

Moreover $N_{5.14}$ is stochastically bounded, uniformly in (n, α, β) .

The proof is more straightforward than that of Proposition 5.13 and is given in Section 7 below.

Lemma 5.15 *For all $n \in \mathbb{N}$, $0 \leq \beta \leq 1/2$ and $0 < d \leq 1$,*

$$a_n^{\beta\gamma} (\sqrt{a_n} \vee d)^{\gamma_1 - 1} \leq d \vee a_n^\beta.$$

Proof. Recall that $\gamma_1 - 1 = \gamma - \frac{1}{2}$ and $2\gamma - \frac{1}{2} > 1$.

Case 1. $d \geq a_n^\beta$.

$$a_n^{\beta\gamma} (\sqrt{a_n} \vee d)^{\gamma_1 - 1} \leq d^{2\gamma - \frac{1}{2}} \leq d.$$

Case 2. $d < a_n^\beta$.

$$a_n^{\beta\gamma} (\sqrt{a_n} \vee d)^{\gamma_1 - 1} \leq a_n^{\beta\gamma + \beta(\gamma - \frac{1}{2})} = a_n^{\beta(2\gamma - \frac{1}{2})} \leq a_n^\beta. \quad \blacksquare$$

We are finally ready to complete the

Proof of Proposition 5.1. Let $0 \leq m \leq \bar{m}$ and assume (P_m) (in (5.5)). We must derive (P_{m+1}) . Let $\varepsilon_0 \in (0, 1)$, $M = \lceil \frac{2}{\varepsilon_0} \rceil$, $\varepsilon_1 = \frac{1}{M} \leq \varepsilon_0/2$ and set $\alpha_i = i\varepsilon_1$ for $i = 0, 1, \dots, M$, so that $\alpha_i \in [\varepsilon_1, 1]$ for $i \geq 1$. Let n, ξ, K , and β be as in (P_m) where we may assume $\xi > 1/2$ without loss of generality. Define $\eta_1 = 1 - \xi \in (0, 1/2)$, $\xi' = \xi + (1 - \xi)/2 \in (\xi, 1)$,

$$N_2(m, n, \xi, \varepsilon_0, K, \beta)(\omega) = \vee_{i=1}^M N_{5.13}(m, n, \eta_1, \varepsilon_0/2, K + 1, \alpha_i, \beta)(\omega),$$

$$N_3(m, n, \xi, \varepsilon_0, K, \beta)(\omega) = \vee_{i=1}^M N_{5.14}(m, n, \eta_1, \varepsilon_0/2, K + 1, \alpha_i, \beta)(\omega),$$

$$\begin{aligned}N_4(m, n, \xi, \varepsilon_0, K, \beta) &= \left\lceil \frac{2}{1 - \xi} ((N_{5.11}(m, n, \eta_1/2, \varepsilon_0/2, K + 1, \beta) \vee N'_{5.13}(\eta_1 \varepsilon_1)) + 1) \right\rceil \\ &\equiv \left\lceil \frac{1}{1 - \xi} N_5(m, n, \eta_1, \varepsilon_0, K, \beta) \right\rceil,\end{aligned}$$

(recall ε_1 is a function of ε_0) and

$$(5.91) \quad N_1(m, n, \xi, \varepsilon_0, K, \beta)(\omega) = (N_2 \vee N_3 \vee N_4(m, n, \xi, \varepsilon_0, K, \beta)) \vee N_1(0, \xi', K) + 1 \in \mathbb{N} \text{ a.s.}$$

Recall that in the verification of (P_0) , we may take $\varepsilon_0 = 0$ and $N_1 = N_1(0, \xi', K)$ was independent of n and β . Then $N_1 = N_1(m, n, \xi, \varepsilon_0, K, \beta)$ is stochastically bounded uniformly in (n, β) because $N_{5.11}$, $N_{5.13}$ and $N_{5.14}$ all are.

Assume

$$N \geq N_1, \quad (t, x) \in Z(N, n, K, \beta), \quad t' \leq T_K, \quad \text{and } d \equiv d((t, x), (t', x')) \leq 2^{-N}.$$

Suppose first that

$$(5.92) \quad a_n > 2^{-N_5(m, n, \eta_1, \varepsilon_0, K, \beta)}.$$

Since $N \geq N_1(0, \xi', K)$, we have by (P_0) , with $\varepsilon_0 = 0$ and ξ' in place of ξ , and the fact that $\tilde{\gamma}_{m+1} - 1 \leq 1$,

$$\begin{aligned} |u(t', x')| &\leq 2^{-N\xi'} \\ &\leq 2^{-N\xi'} \left[(\sqrt{a_n} \vee 2^{-N})^{\tilde{\gamma}_{m+1}-1} \right] 2^{N_5/2} \\ &\leq 2^{-N(1-\xi)/2} 2^{N_5/2} 2^{-N\xi} \left[(\sqrt{a_n} \vee 2^{-N})^{\tilde{\gamma}_{m+1}-1} + a_n^\beta \right] \\ &\leq 2^{-N\xi} \left[(\sqrt{a_n} \vee 2^{-N})^{\tilde{\gamma}_{m+1}-1} + a_n^\beta \right], \end{aligned}$$

where in the last line we used $N \geq N_4 \geq (1 - \xi)^{-1} N_5$. This proves (P_{m+1}) .

So assume now that

$$(5.93) \quad a_n \leq 2^{-N_5(m, n, \eta_1, \varepsilon_0, K, \beta)}.$$

Let $N' = N - 1 \geq N_2 \vee N_3$. Note that (\hat{t}_0, \hat{x}_0) (the point near (t, x) in the definition of $Z(N, n, K, \beta)$) is in $Z(N, n, K + 1, \beta) \subset Z(N', n, K + 1, \beta)$ and by the triangle inequality $d((\hat{t}_0, \hat{x}_0), (t', x')) \leq 2^{-N'}$. (5.93) shows that (5.70) holds with $(\varepsilon_0/2, K + 1)$ in place of (ε_0, K) . Therefore the inequality $N' \geq N_2$ allows us to apply the conclusion of Proposition 5.13 for $\alpha = \alpha_i \geq \varepsilon_1$, $i = 1, \dots, M$ with (\hat{t}_0, \hat{x}_0) in place of (t, x) , $\varepsilon_0/2$ in place of ε_0 , and N' in place of N . Simpler reasoning, using $N' \geq N_3$, allows us to apply the conclusion of Proposition 5.14 with the same parameter values.

Choose $i \in \{1, \dots, M\}$ so that

$$(5.94) \quad \begin{aligned} (i) \text{ if } 2^{-N'} > \sqrt{a_n}, \text{ then } a_n^{\frac{\alpha_i}{2}} < 2^{-N'} \leq a_n^{\frac{\alpha_i-1}{2}} = a_n^{\frac{\alpha_i}{2}} a_n^{-\frac{\varepsilon_1}{2}}, \\ (ii) \text{ if } 2^{-N'} \leq \sqrt{a_n}, \text{ then } i = M \text{ and so } a_n^{\frac{\alpha_i}{2}} = \sqrt{a_n} \geq 2^{-N'}. \end{aligned}$$

In either case we have

$$(5.95) \quad a_n^{\frac{\alpha_i}{2}} \vee 2^{-N'} \leq \sqrt{a_n} \vee 2^{-N'},$$

and

$$(5.96) \quad a_n^{-\frac{3\alpha_i}{4}} (\sqrt{a_n} \vee 2^{-N'})^{3/2} \leq a_n^{-\frac{3\varepsilon_1}{4}}.$$

Now apply Propositions 5.13 and 5.14, as described above, as well as (5.95), and the facts that $\tilde{\gamma}_m = \gamma_m$ for $m \leq \bar{m}$, $\gamma_{m+1} = \gamma\gamma_m + \frac{1}{2}$ and $d((\hat{t}_0, \hat{x}_0), (t', x')) \leq 2^{-N'}$, to conclude

$$\begin{aligned}
& |u(\hat{t}_0, \hat{x}_0) - u(t', x')| \\
& \leq |u_{1, a_n^{\alpha_i}}(\hat{t}_0, \hat{x}_0) - u_{1, a_n^{\alpha_i}}(t', x')| + |u_{2, a_n^{\alpha_i}}(\hat{t}_0, \hat{x}_0) - u_{2, a_n^{\alpha_i}}(t', x')| \\
& \leq 2^{-89} a_n^{-\frac{\varepsilon_0}{2}} \left\{ 2^{-N'\xi} \left[a_n^\beta + a_n^{-\frac{3\alpha_i}{4}} a_n^{\beta\gamma} (\sqrt{a_n} \vee 2^{-N'})^{(\gamma+1)} + a_n^{-\frac{3\alpha_i}{4}} (\sqrt{a_n} \vee 2^{-N'})^{(\gamma\gamma_m+1)} \right. \right. \\
& \quad \left. \left. + 1(m = \bar{m})(\sqrt{a_n} \vee 2^{-N'}) \right] + 2^{-N'(\frac{\xi}{2} + \gamma)} \left[(\sqrt{a_n} \vee 2^{-N'})^{\gamma(\gamma_m-1)} + a_n^{\beta\gamma} \right] \right. \\
& \quad \left. + 2^{-N'\xi} \left[(\sqrt{a_n} \vee 2^{-N'})^{(\gamma_{m+1}-1)} + a_n^{\beta\gamma} (\sqrt{a_n} \vee 2^{-N'})^{(\gamma-\frac{1}{2})} \right] \right\} \\
(5.97) \quad & \leq 2^{-89} a_n^{-\frac{\varepsilon_0}{2}} 2^{-N'\xi} \left\{ a_n^\beta + a_n^{-\frac{3\alpha_i}{4}} (\sqrt{a_n} \vee 2^{-N'})^{\frac{3}{2}} a_n^{\beta\gamma} (\sqrt{a_n} \vee 2^{-N'})^{(\gamma-\frac{1}{2})} \right. \\
& \quad \left. + a_n^{-\frac{3\alpha_i}{4}} (\sqrt{a_n} \vee 2^{-N'})^{\frac{3}{2}} (\sqrt{a_n} \vee 2^{-N'})^{(\gamma_{m+1}-1)} + 1(m = \bar{m})(\sqrt{a_n} \vee 2^{-N'}) \right. \\
& \quad \left. + (\sqrt{a_n} \vee 2^{-N'})^{(\gamma-\frac{1}{2})} \left[(\sqrt{a_n} \vee 2^{-N'})^{\gamma(\gamma_m-1)} + a_n^{\beta\gamma} \right] \right. \\
& \quad \left. + (\sqrt{a_n} \vee 2^{-N'})^{(\gamma_{m+1}-1)} + a_n^{\beta\gamma} (\sqrt{a_n} \vee 2^{-N'})^{(\gamma-\frac{1}{2})} \right\}.
\end{aligned}$$

Now apply (5.96) and combine some duplicate terms to bound $|u(\hat{t}_0, \hat{x}_0) - u(t', x')|$ by

$$2^{-87} a_n^{-\frac{\varepsilon_0}{2} - \frac{3\varepsilon_1}{4}} 2^{-N'\xi} \left[a_n^\beta + a_n^{\beta\gamma} (\sqrt{a_n} \vee 2^{-N'})^{\gamma-\frac{1}{2}} + (\sqrt{a_n} \vee 2^{-N'})^{(\gamma_{m+1}-1)} + 1(m = \bar{m})(\sqrt{a_n} \vee 2^{-N'}) \right].$$

Use the fact that

$$(\sqrt{a_n} \vee 2^{-N'})^{(\gamma_{m+1}-1)} + 1(m = \bar{m})(\sqrt{a_n} \vee 2^{-N'}) \leq 2(\sqrt{a_n} \vee 2^{-N'})^{((\gamma_{m+1}\wedge 2)-1)}$$

(consider $m < \bar{m}$ and $m = \bar{m}$ separately) and $\varepsilon_1 \leq \varepsilon_0/2$ in the above to derive

$$\begin{aligned}
(5.98) \quad & |u(\hat{t}_0, \hat{x}_0) - u(t', x')| \leq 2^{-86} a_n^{-\varepsilon_0} 2^{-N'\xi} \left[a_n^\beta + a_n^{\beta\gamma} (\sqrt{a_n} \vee 2^{-N'})^{(\gamma-\frac{1}{2})} + (\sqrt{a_n} \vee 2^{-N'})^{((\gamma_{m+1}\wedge 2)-1)} \right] \\
& \leq 2^{-84} a_n^{-\varepsilon_0} 2^{-N\xi} \left[a_n^\beta + a_n^{\beta\gamma} (\sqrt{a_n} \vee 2^{-N})^{(\gamma-\frac{1}{2})} + (\sqrt{a_n} \vee 2^{-N})^{((\gamma_{m+1}\wedge 2)-1)} \right].
\end{aligned}$$

Finally combine (5.98) and $|u(\hat{t}_0, \hat{x}_0)| \leq \sqrt{a_n} 2^{-N}$ to conclude (note also that $\gamma_0 = 1$ so the $k = 0$ term in the sum below simplifies considerably)

$$\begin{aligned}
(5.99) \quad & |u(t', x')| \leq \sqrt{a_n} 2^{-N} + 2^{-84} a_n^{-\varepsilon_0} 2^{-N\xi} \left[\sum_{k=0}^1 a_n^{\beta\gamma^k} (2^{-N} \vee \sqrt{a_n})^{\gamma^k-1} + (2^{-N} \vee \sqrt{a_n})^{\tilde{\gamma}_{m+1}-1} \right] \\
& \leq a_n^{-\varepsilon_0} 2^{-N\xi} \left[\sqrt{a_n} 2^{-N(1-\xi)} + 2^{-84} \left[\sum_{k=0}^1 a_n^{\beta\gamma^k} (2^{-N} \vee \sqrt{a_n})^{\gamma^k-1} + (2^{-N} \vee \sqrt{a_n})^{\tilde{\gamma}_{m+1}-1} \right] \right].
\end{aligned}$$

Our definition of N_1 (and especially N_4) ensures that $N(1-\xi) \geq 1$ and hence

$$\sqrt{a_n} 2^{-N(1-\xi)} \leq \frac{\sqrt{a_n}}{2} \leq \frac{a_n^\beta}{2}.$$

In addition by Lemma 5.15

$$a_n^{\beta\gamma} (\sqrt{a_n} \vee 2^{-N})^{\gamma_1-1} \leq a_n^\beta \vee 2^{-N} \leq a_n^\beta + (2^{-N} \vee \sqrt{a_n})^{\tilde{\gamma}_{m+1}-1}.$$

Substitute the last bounds into (5.99) to obtain (P_{m+1}) and hence complete the induction. \blacksquare

6 Proof of Proposition 3.3

We continue to assume $b \equiv 0$ in this Section. Having established the bound $(P_{\overline{m}+1})$ in Proposition 5.1, we are now free to use the conclusions of Corollary 5.9, Proposition 5.11 and Proposition 5.14, with $m = \overline{m} + 1$, to derive local moduli of continuity for u'_{1,a_n^α} and u_{2,a_n^α} . In view of our main goal, Proposition 3.3, it is the space modulus we will need—the time modulus was only needed to carry out the induction leading to $(P_{\overline{m}+1})$.

We fix a $K_0 \in \mathbb{N}^{\geq K_1}$ and positive constants $\varepsilon_0, \varepsilon_1$ as in (3.10). For $M, n \in \mathbb{N}$ and $0 < \beta \leq \frac{1}{2} - \varepsilon_1$, define

$$(6.1) \quad \alpha = \alpha(\beta) = 2(\beta + \varepsilon_1) \in [0, 1].$$

and

$$\begin{aligned} U_{M,n,\beta}^{(1)} = \inf \left\{ t : \text{there are } \varepsilon \in [0, 2^{-M}], |x| \leq K_0 + 1, \hat{x}_0, x' \in \mathbb{R}, \text{ s.t. } |x - x'| \leq 2^{-M}, \right. \\ |x - \hat{x}_0| \leq \varepsilon, |u(t, \hat{x}_0)| \leq a_n \wedge (\sqrt{a_n} \varepsilon), |u'_{1,a_n}(t, \hat{x}_0)| \leq a_n^\beta, \text{ and} \\ \left. |u'_{1,a_n^\alpha}(t, x) - u'_{1,a_n^\alpha}(t, x')| > 2^{-82} a_n^{-\varepsilon_0 - \frac{3\varepsilon_1}{2}} |x - x'|^{1-\varepsilon_0} \left[a_n^{-3\beta/2} (\varepsilon \vee |x' - x|)^{2\gamma} + 1 \right. \right. \\ \left. \left. + a_n^{\beta(\gamma - \frac{3}{2})} (\varepsilon \vee |x' - x|)^\gamma \right] \right\} \wedge T_{K_0}. \end{aligned}$$

Define $U_{M,n,0}^{(1)}$ by the same expression (with $\beta = 0$) but with the condition on $|u'_{1,a_n}(t, \hat{x}_0)|$ omitted.

$\{U_{M,n,\beta}^{(1)} < t\}$ is the projection onto Ω of a Borel $\times \mathcal{F}_t$ -measurable set in $K \times [0, t] \times \Omega$ where K is a compact subset of \mathbb{R}^4 , and so $U_{M,n,\beta}^{(1)}$ is an (\mathcal{F}_t) -stopping time as in IV.T52 of (M66).

Lemma 6.1 *For each $n \in \mathbb{N}$ and β as in (3.12), $U_{M,n,\beta}^{(1)} \uparrow T_{K_0}$ as $M \uparrow \infty$, and in fact*

$$\lim_{M \rightarrow \infty} \sup_{n, 0 \leq \beta \leq \frac{1}{2} - \varepsilon_1} P(U_{M,n,\beta}^{(1)} < T_{K_0}) = 0.$$

Proof. By monotonicity in M the first assertion is immediate from the second. Proposition 5.1 allows us to apply Corollary 5.9 with $m = \overline{m} + 1$, $\eta_1 = \varepsilon_0$, $K = K_0 + 1$, and α, β as in (6.1) and (3.12), respectively. Hence there is an $N_0 = N_0(n, \varepsilon_0, \varepsilon_1, K_0 + 1, \beta) \in \mathbb{N}$ a.s., stochastically bounded uniformly in (n, β) (as in (3.12)), and such that if

$$(6.2) \quad N \geq N_0(\omega), (t, x) \in Z(N, n, K_0 + 1, \beta), |x - x'| \leq 2^{-N},$$

then

$$\begin{aligned} & |u'_{1,a_n^\alpha}(t, x) - u'_{1,a_n^\alpha}(t, x')| \\ & \leq 2^{-85} |x - x'|^{1-\varepsilon_0} a_n^{-\varepsilon_0 - \frac{3\varepsilon_1}{2}} \left[a_n^{-\frac{3\beta}{2}} 2^{-2N\gamma} + 1 + a_n^{\beta(\gamma - \frac{3}{2})} (a_n^{\beta + \varepsilon_1} \vee 2^{-N})^\gamma \right]. \end{aligned}$$

Note that $a_n^{-\frac{3\beta}{2} + \beta\gamma + (\beta + \varepsilon_1)\gamma} \leq 1$ since $\gamma > 3/4$, and so by the above we have

$$(6.3) \quad \begin{aligned} & |u'_{1,a_n^\alpha}(t, x) - u'_{1,a_n^\alpha}(t, x')| \\ & \leq 2^{-84} a_n^{-\varepsilon_0 - \frac{3\varepsilon_1}{2}} |x - x'|^{1-\varepsilon_0} \left[a_n^{-\frac{3\beta}{2}} 2^{-2N\gamma} + 1 + a_n^{\beta(\gamma - \frac{3}{2})} 2^{-N\gamma} \right]. \end{aligned}$$

Let us assume $\beta > 0$ for if $\beta = 0$ we can just omit the bound on $|u'_{1,a_n^\alpha}(t, \hat{x}_0)|$ in what follows. Assume $M \geq N_0(n, \varepsilon_0, \varepsilon_1, K_0 + 1, \beta)$. Suppose for some $t < T_{K_0} (\leq T_{K_0+1})$ there are $\varepsilon \in [0, 2^{-M}]$, $|x| \leq K_0 + 1$, $\hat{x}_0, x' \in \mathbb{R}$ satisfying $|x - x'| \leq 2^{-M}$, $|\hat{x}_0 - x| \leq \varepsilon$, $|u(t, \hat{x}_0)| \leq a_n \wedge (\sqrt{a_n} \varepsilon)$, and $|u'_{1,a_n}(t, \hat{x}_0)| \leq a_n^\beta$. If $x \neq x'$, then $0 < |x - x'| \vee \varepsilon \leq 2^{-M} \leq 2^{-N_0}$ and we may choose $N \geq N_0$ so that $2^{-N-1} < \varepsilon \vee |x - x'| \leq 2^{-N}$. Then (6.2) holds and so by (6.3),

$$\begin{aligned} |u'_{1,a_n^\alpha}(t, x) - u'_{1,a_n^\alpha}(t, x')| &\leq 2^{-82} a_n^{-\varepsilon_0 - \frac{3\varepsilon_1}{2}} |x - x'|^{1-\varepsilon_0} \left[a_n^{-\frac{3\beta}{2}} (\varepsilon \vee |x - x'|)^{2\gamma} + 1 \right. \\ &\quad \left. + a_n^{\beta(\gamma - \frac{3}{2})} (\varepsilon \vee |x - x'|)^\gamma \right]. \end{aligned}$$

If $x = x'$ the above is trivial. This implies $U_{M,n,\beta}^{(1)} = T_{K_0}$ by its definition. We have therefore shown

$$P(U_{M,n,\beta}^{(1)} < T_{K_0}) \leq P(M < N_0).$$

This completes the proof because $N_0(n, \varepsilon_0, \varepsilon_1, K_0 + 1, \beta)$ is stochastically bounded uniformly in (n, β) (as in (3.12)). \blacksquare

Turning next to u_{2,a_n^α} , for $0 < \beta \leq \frac{1}{2} - \varepsilon_1$, define

$$\begin{aligned} U_{M,n,\beta}^{(2)} &= \inf \left\{ t : \text{there are } \varepsilon \in [0, 2^{-M}], |x| \leq K_0 + 1, \hat{x}_0, x' \in \mathbb{R}, \text{ s.t. } |x - x'| \leq 2^{-M}, \right. \\ &\quad |x - \hat{x}_0| \leq \varepsilon, |u(t, \hat{x}_0)| \leq a_n \wedge (\sqrt{a_n} \varepsilon), |u'_{1,a_n}(t, \hat{x}_0)| \leq a_n^\beta, \text{ and} \\ &\quad |u_{2,a_n^\alpha}(t, x) - u_{2,a_n^\alpha}(t, x')| > 2^{-87} a_n^{-\varepsilon_0} \left(|x - x'|^{\frac{1-\varepsilon_0}{2}} \left[(\sqrt{a_n} \vee \varepsilon \vee |x' - x|)^{2\gamma} \right. \right. \\ &\quad \left. \left. + a_n^{\beta\gamma} (\sqrt{a_n} \vee \varepsilon \vee |x' - x|)^\gamma \right] \right. \\ &\quad \left. \left. + |x - x'|^{1-\varepsilon_0} a_n^{\beta + \frac{\varepsilon_1}{4}} \right) \right\} \wedge T_{K_0}. \end{aligned}$$

Define $U_{M,n,0}^{(2)}$ by the same expression (with $\beta = 0$) but with the condition on $|u'_{1,a_n}(t, \hat{x}_0)|$ omitted. Just as for $U^{(1)}$, $U_{M,n,\beta}^{(2)}$ is an \mathcal{F}_t -stopping time.

Lemma 6.2 *For each $n \in \mathbb{N}$ and β as in (3.12), $U_{M,n,\beta}^{(2)} \uparrow T_{K_0}$ as $M \uparrow \infty$, and in fact*

$$\lim_{M \rightarrow \infty} \sup_{n, 0 \leq \beta \leq \frac{1}{2} - \varepsilon_1} P(U_{M,n,\beta}^{(2)} < T_{K_0}) = 0.$$

Proof. As before we only need to show the second assertion. Proposition 5.1 allows us to apply Proposition 5.14 with $m = \bar{m} + 1$, $\eta_1 = \varepsilon_0$, $K = K_0 + 1$, and α, β as in (3.12), (6.1). Hence there is an $N_0 = N_0(n, \varepsilon_0, \varepsilon_1, K_0 + 1, \beta) \in \mathbb{N}$ a.s., stochastically bounded uniformly in (n, β) (as in (3.12)), and such that if

$$(6.4) \quad N \geq N_0(\omega), \quad (t, x) \in Z(N, n, K_0 + 1, \beta), \quad |x - x'| \leq 2^{-N},$$

then

$$\begin{aligned} &|u_{2,a_n^\alpha}(t, x) - u_{2,a_n^\alpha}(t, x')| \\ &\leq 2^{-89} a_n^{-\varepsilon_0} \left\{ |x - x'|^{\frac{1-\varepsilon_0}{2}} \left[(\sqrt{a_n} \vee 2^{-N})^{2\gamma} + a_n^{\beta\gamma} (\sqrt{a_n} \vee 2^{-N})^\gamma \right] \right. \\ (6.5) \quad &\quad \left. + |x - x'|^{1-\varepsilon_0} \left[a_n^{(\beta+\varepsilon_1)(2\gamma-\frac{1}{2})} + a_n^{\beta\gamma} a_n^{(\beta+\varepsilon_1)(\gamma-\frac{1}{2})} \right] \right\}. \end{aligned}$$

Since $\gamma > \frac{3}{4}$,

$$a_n^{(\beta+\varepsilon_1)(2\gamma-\frac{1}{2})} + a_n^{\beta\gamma} a_n^{(\beta+\varepsilon_1)(\gamma-\frac{1}{2})} \leq a_n^{\beta+\varepsilon_1} + a_n^{\beta+\frac{\varepsilon_1}{4}} \leq 2a_n^{\beta+\frac{\varepsilon_1}{4}}.$$

Therefore (6.5) shows that (6.4) implies

$$(6.6) \quad \begin{aligned} & |u_{2,a_n^\alpha}(t, x) - u_{2,a_n^\alpha}(t, x')| \\ & \leq 2^{-89} a_n^{-\varepsilon_0} \left\{ |x - x'|^{\frac{1-\varepsilon_0}{2}} \left[(\sqrt{a_n} \vee 2^{-N})^{2\gamma} + a_n^{\beta\gamma} (\sqrt{a_n} \vee 2^{-N})^\gamma \right] \right. \\ & \quad \left. + 2|x - x'|^{1-\varepsilon_0} a_n^{\beta+\frac{\varepsilon_1}{4}} \right\}. \end{aligned}$$

The proof is now completed just as for Lemma 6.1 where (6.6) is used in place of (6.3). ■

Notation.

$$(6.7) \quad \tilde{\Delta}_{u'_1}(n, \varepsilon, \varepsilon_0, \beta) = a_n^{-\varepsilon_0} \varepsilon^{-\varepsilon_0} \left\{ \varepsilon + (\varepsilon a_n^{-3/4} + a_n^{-1/4}) \left(\varepsilon^{2\gamma} + a_n^{\beta\gamma} (\varepsilon \vee \sqrt{a_n})^\gamma \right) \right\}.$$

For $0 < \beta \leq \frac{1}{2} - \varepsilon_1$, define

$$(6.8) \quad \begin{aligned} U_{M,n,\beta}^{(3)} = \inf \left\{ t : \text{there are } \varepsilon \in [2^{-a_n^{-(\beta+\varepsilon_1)\varepsilon_0/4}}, 2^{-M}], |x| \leq K_0 + 1, \hat{x}_0 \in \mathbb{R}, s.t. \right. \\ |x - \hat{x}_0| \leq \varepsilon, |u(t, \hat{x}_0)| \leq a_n \wedge (\sqrt{a_n} \varepsilon), |u'_{1,a_n}(t, \hat{x}_0)| \leq a_n^\beta, \text{ and} \\ \left. |u'_{1,a_n}(t, x) - u'_{1,a_n^\alpha}(t, x)| > 2^{-78} (\tilde{\Delta}_{u'_1}(n, \varepsilon, \varepsilon_0, \beta) + a_n^{\beta+\frac{\varepsilon_1}{8}}) \right\} \wedge T_{K_0}. \end{aligned}$$

Define $U_{M,n,0}^{(3)}$ by the same expression (with $\beta = 0$) but with the condition on $|u'_{1,a_n^\alpha}(t, \hat{x}_0)|$ omitted. Just as for $U^{(1)}$, $U_{M,n,\beta}^{(3)}$ is an \mathcal{F}_t -stopping time.

Lemma 6.3 *For each $n \in \mathbb{N}$ and β as in (3.12), $U_{M,n,\beta}^{(3)} \uparrow T_{K_0}$ as $M \uparrow \infty$, and in fact*

$$\lim_{M \rightarrow \infty} \sup_{n, 0 \leq \beta \leq \frac{1}{2} - \varepsilon_1} P(U_{M,n,\beta}^{(3)} < T_{K_0}) = 0.$$

Proof. It suffices to prove the second assertion. By Proposition 5.1 we may apply Proposition 5.11 with $m = \bar{m} + 1$, $\eta_1 = \varepsilon_0$, $K = K_0 + 1$ and β as in (3.12). Note also that if $s = t - a_n^\alpha + a_n$, then

$$(6.8) \quad \sqrt{t - s} \leq a_n^{\alpha/2} = a_n^{\beta+\varepsilon_1},$$

and

$$(6.9) \quad a_n^{\beta+\varepsilon_1} \leq N^{-4/\varepsilon_0} \iff 2^{-N} \geq 2^{-a_n^{-(\beta+\varepsilon_1)\varepsilon_0/4}}.$$

So Proposition 5.11 shows there is an $N_0 = N_0(n, \varepsilon_0, K_0 + 1, \beta) \in \mathbb{N}$ a.s., stochastically bounded uniformly in (n, β) , and so that if

$$(6.10) \quad N \geq N_0, (t, x) \in Z(N, n, K_0 + 1, \beta) \text{ and } 2^{-N} \geq 2^{-a_n^{-(\beta+\varepsilon_1)\varepsilon_0/4}},$$

then

$$\begin{aligned}
& |u'_{1,a_n}(t, x) - u'_{1,a_n^\alpha}(t, x)| \\
&= |F_{a_n}(t - a_n^\alpha + a_n, t, x) - F_{a_n}(t, t, x)| \quad (\text{by (5.47)}) \\
(6.11) \quad &\leq 2^{-81} a_n^{-\varepsilon_0} \left\{ 2^{N\varepsilon_0} \left[2^{-N} + a_n^{-1/4} (2^{-N} a_n^{-1/2} + 1) (2^{-2\gamma N} + a_n^{\beta\gamma} (2^{-N} \vee \sqrt{a_n})^\gamma) \right] \right. \\
&\quad \left. + a_n^{(\beta+\varepsilon_1)(1-\varepsilon_0)} \left[a_n^{(\beta+\varepsilon_1)(2\gamma-\frac{3}{2})} + a_n^{\beta\gamma} a_n^{(\beta+\varepsilon_1)(\gamma-\frac{3}{2})} \right] \right\}.
\end{aligned}$$

We have used $\beta + \varepsilon_1 \leq \frac{1}{2}$ (from (3.12)) in the last line. The fact that

$$(\beta + \varepsilon_1)(2\gamma - \frac{3}{2}) > \beta\gamma + (\beta + \varepsilon_1)(\gamma - \frac{3}{2})$$

implies that

$$\begin{aligned}
& a_n^{-\varepsilon_0} a_n^{(\beta+\varepsilon_1)(1-\varepsilon_0)} \left[a_n^{(\beta+\varepsilon_1)(2\gamma-\frac{3}{2})} + a_n^{\beta\gamma+(\beta+\varepsilon_1)(\gamma-\frac{3}{2})} \right] \\
&\leq 2 a_n^{-\varepsilon_0} a_n^{(\beta+\varepsilon_1)(1-\varepsilon_0)+\beta(2\gamma-\frac{3}{2})+\varepsilon_1(\gamma-\frac{3}{2})} \\
&\leq 2 a_n^{\beta(2\gamma-\frac{1}{2}-\varepsilon_0)+\varepsilon_1(\gamma-\frac{1}{2}-\varepsilon_0)-\frac{\varepsilon_1}{100}} \\
&\leq 2 a_n^{\beta+\frac{\varepsilon_1}{8}},
\end{aligned}$$

where (3.10) is used in the last two inequalities. This allows us to simplify (6.11) and show that (6.10) implies

$$(6.12) \quad |u'_{1,a_n}(t, x) - u'_{1,a_n^\alpha}(t, x)| \leq 2^{-81} [\tilde{\Delta}_{u'_1, \varepsilon_0}(n, 2^{-N}, \varepsilon_0, \beta) + 2 a_n^{\beta+\frac{\varepsilon_1}{8}}].$$

The proof now is similar to that of Lemma 6.1. As before, we may assume $\beta > 0$. Assume $M \geq N_0(n, \varepsilon_0, K_0 + 1, \beta)$. Suppose for some $t < T_{K_0}$ there are $\varepsilon \in [2^{-a_n^{-(\beta+\varepsilon_1)\varepsilon_0/4}}, 2^{-M}]$, $|x| \leq K_0 + 1$, and $\hat{x}_0 \in \mathbb{R}$, such that $|\hat{x}_0 - x| \leq \varepsilon$, $|u(t, \hat{x}_0)| \leq a_n \wedge (\sqrt{a_n}\varepsilon)$, and $|u'_{1,a_n}(t, \hat{x}_0)| \leq a_n^\beta$. We may choose $N \geq M \geq N_0(\omega)$ so that $2^{-N-1} < \varepsilon \leq 2^{-N}$. Then $(t, x) \in Z(N, n, K_0 + 1, \beta)$ and $2^{-N} \geq \varepsilon \geq 2^{-a_n^{-(\beta+\varepsilon_1)\varepsilon_0/4}}$, and therefore (6.10) holds. Therefore we may use (6.12) and the fact that $\tilde{\Delta}(n, 2\varepsilon, \varepsilon_0, \beta) \leq 8\tilde{\Delta}(n, \varepsilon, \varepsilon_0, \beta)$ to see that

$$|u'_{1,a_n}(t, x) - u'_{1,a_n^\alpha}(t, x)| \leq 2^{-78} [\tilde{\Delta}_{u'_1}(n, \varepsilon, \varepsilon_0, \beta) + a_n^{\beta+\frac{\varepsilon_1}{8}}].$$

This shows that $M \geq N_0(n, \varepsilon_0, K_0 + 1, \beta)$ implies $U_{M,n,\beta}^{(3)} = T_{K_0}$ and so

$$\sup_{n, 0 \leq \beta \leq \frac{1}{2} - \varepsilon_1} P(U_{M,n,\beta}^{(3)} < T_{K_0}) \leq \sup_{n, 0 \leq \beta \leq \frac{1}{2} - \varepsilon_1} P(N_0 > M) \rightarrow 0 \text{ as } M \rightarrow \infty$$

by the stochastic boundedness of N_0 uniformly in (n, β) . ■

Finally for $M \in \mathbb{N}$, define

$$\begin{aligned}
U_M^{(4)} = \inf \left\{ t : \text{there are } \varepsilon \in [0, 2^{-M}], |x| \leq K_0 + 1, \hat{x}_0, x' \in \mathbb{R}, \text{ s.t. } |x - x'| \leq 2^{-M}, \right. \\
\left. |x - \hat{x}_0| \leq \varepsilon, |u(t, \hat{x}_0)| \leq \varepsilon, \text{ and } |u(t, x) - u(t, x')| > (\varepsilon \vee |x' - x|)^{1-\varepsilon_0} \right\} \wedge T_{K_0}.
\end{aligned}$$

Lemma 6.4 $U_M^{(4)} \uparrow T_{K_0}$ as $M \uparrow \infty$, and

$$\lim_{M \rightarrow \infty} P(U_M^{(4)} < T_{K_0}) = 0.$$

Proof. It suffices to prove the second result. This follows easily from Theorem 2.3 as in the proof of Lemma 6.1. The constant multiplicative factors arising in the proof can easily be handled by applying Theorem 2.3 with $\xi = 1 - \varepsilon_0/2$ in place of $\xi = 1 - \varepsilon_0$. \blacksquare

Let

$$U_{M,n,\beta} = \wedge_{j=1}^3 U_{M,n,\beta}^{(j)},$$

and

$$(6.13) \quad U_{M,n} = \left(\wedge_{i=0}^{L(\varepsilon_0, \varepsilon_1)} U_{M,n,\beta_i} \right) \wedge U_M^{(4)},$$

where we recall that $\{\beta_i : i \leq L\}$ were introduced in (3.11). We have suppressed the dependence of $U_{M,n}$ on our fixed values of K_0, ε_0 and ε_1 . Note that $\beta_i \in [0, \frac{1}{2} - \varepsilon_1]$ for $i = 0, \dots, L$ by (3.12) and $\alpha_i = \alpha(\beta_i)$. Lemmas 6.1, 6.2, 6.3 and 6.4 therefore show that $\{U_{M,n}\}$ satisfy hypothesis (H_1) of Proposition 2.1. Hence to complete the proof of Proposition 3.3 it suffices to establish compactness of $\tilde{J}_{n,i}(s)$, and the inclusion $\tilde{J}_{n,i}(s) \supset J_{n,i}(s)$ for all $s < U_{M,n}$, (n, M) as in (3.14), and $i = 0, \dots, L$. The next lemmas will show the inclusion part of the proof. We assume (n, M) satisfies (3.14) throughout the rest of this Section.

Lemma 6.5 *If $i \in \{0, \dots, L\}$, $0 \leq s < U_{M,n}$, and $x \in J_{n,i}(s)$, then (a) $|u'_{1,a_n}(s, \hat{x}_n(s, x)) - u'_{1,a_n^{\alpha_i}}(s, \hat{x}_n(s, x))| \leq 2^{-74} a_n^{\beta_i + \frac{\varepsilon_1}{8}}$, (b) for $i > 0$, $|u'_{1,a_n^{\alpha_i}}(s, \hat{x}_n(s, x))| \leq a_n^{\beta_i}/2$, (c) for $i < L$, $u'_{1,a_n^{\alpha_i}}(s, \hat{x}_n(s, x)) \geq a_n^{\beta_i+1}/8$.*

Proof. (a) Assume (n, i, s, x) are as above and set $\varepsilon = \sqrt{a_n}$. We have $|\langle u_s, \Phi_x^{m_{n+1}} \rangle| \leq a_n$ and

$$\text{Supp}(\Phi_x^{m_{n+1}}) \subset [x - \sqrt{a_n}, x + \sqrt{a_n}].$$

Using the continuity of $u(s, \cdot)$, we conclude that

$$(6.14) \quad |u(s, \hat{x}_n(s, x))| \leq a_n = a_n \wedge (\sqrt{a_n} \varepsilon), \quad |\hat{x}_n(s, x) - x| \leq \varepsilon.$$

The definition of $J_{n,i}$ also implies

$$(6.15) \quad |u'_{1,a_n}(s, \hat{x}_n(s, x))| \leq a_n^{\beta_i}/4 \quad \text{for } i > 0.$$

In addition, (3.14) and $\varepsilon_1 < 1/2$ (by(3.10)) imply

$$(6.16) \quad 2^{-M} \geq \sqrt{a_n} = \varepsilon \geq 2^{-a_n^{-\varepsilon_0 \varepsilon_1/4}}.$$

Combine (6.14), (6.15) and (6.16) with $|\hat{x}_n(s, x)| \leq K_0 + 1$ and $s < U_{M,n} \leq U_{M,n,\beta_i}^{(3)}$, and take $x = \hat{x}_0 = \hat{x}_n(s, x)$ in the definition of $U^{(3)}$, to conclude that

$$(6.17) \quad |u'_{1,a_n}(s, \hat{x}_n(s, x)) - u'_{1,a_n^{\alpha_i}}(s, \hat{x}_n(s, x))| \leq 2^{-78} (\tilde{\Delta}_{u'_1}(n, \sqrt{a_n}, \varepsilon_0, \beta_i) + a_n^{\beta_i + \frac{\varepsilon_1}{8}}).$$

Now

$$\begin{aligned}
\tilde{\Delta}_{u'_1}(n, \sqrt{a_n}, \varepsilon_0, \beta_i) &\leq a_n^{-3\varepsilon_0/2} \left[\sqrt{a_n} + 2a_n^{-1/4} (a_n^\gamma + a_n^{\gamma(\beta_i + \frac{1}{2})}) \right] \\
&\leq 4a_n^{-3\varepsilon_0/2} \left[\sqrt{a_n} + a_n^{\gamma(\beta_i + \frac{1}{2}) - \frac{1}{4}} \right] \quad (\text{since } \beta_i < \frac{1}{2}) \\
&\leq 4 \left[a_n^{\frac{1-3\varepsilon_0}{2}} + a_n^{\beta_i - \frac{3\varepsilon_0}{2} + \frac{3\varepsilon_1}{2}} \right] \quad (\text{using } \gamma > \frac{3}{4} \text{ and } \beta_i \leq \frac{1}{2} - 6\varepsilon_1) \\
&\leq 8a_n^{\beta_i + \frac{\varepsilon_1}{8}}.
\end{aligned}$$

The last line follows from (3.10) and a bit of arithmetic. Use the above bound in (6.17) and conclude that

$$|u'_{1,a_n}(s, \hat{x}_n(s, x)) - u'_{1,a_n^{\alpha_i}}(s, \hat{x}_n(s, x))| \leq 2^{-74} a_n^{\beta_i + \frac{\varepsilon_1}{8}}.$$

(b) This is immediate from (a) and the fact that $|u'_{1,a_n}(s, \hat{x}_n(s, x))| \leq a_n^{\beta_i}/4$ (by the definition of $J_{n,i}$ for $i > 0$).

(c) Since $\varepsilon_0 \leq \varepsilon_1/8$ by (3.10), $a_n^{\beta_i + \frac{\varepsilon_1}{8}} \leq a_n^{\beta_i + 1}$. For $i < L$ we have $u'_{1,a_n}(s, \hat{x}_n(s, x)) \geq a_n^{\beta_i + 1}/4$. The result is now clear from (a) and the triangle inequality. \blacksquare

Lemma 6.6 *If $i \in \{0, \dots, L\}$, $0 \leq s < U_{M,n}$, $x \in J_{n,i}(s)$ and $|x - x'| \leq 5\bar{l}_n(\beta_i)$, then*

- (a) *for $i > 0$, $|u'_{1,a_n^{\alpha_i}}(s, x')| \leq a_n^{\beta_i}$,*
- (b) *for $i < L$, $u'_{1,a_n^{\alpha_i}}(s, x') \geq a_n^{\beta_i + 1}/16$.*

Proof. Let (n, i, s, x, x') be as above and set $\varepsilon = |x - x'| + \sqrt{a_n}$. Then we have (6.14), (6.15), and also (by (3.14))

$$(6.18) \quad \varepsilon \leq 5a_n^{5\varepsilon_1} + \sqrt{a_n} \leq 2^{-M},$$

and

$$(6.19) \quad |x' - \hat{x}_n(s, x)| \leq \varepsilon \leq 2^{-M}, \quad |\hat{x}_n(s, x)| \leq |x| + 1 \leq K_0 + 1.$$

(6.14), (6.15), (6.18) and (6.19) allow us to use $s < U_{M,n} \leq U_{M,n,\beta_i}^{(1)}$ and the definition of $U_{M,n,\beta_i}^{(1)}$ (for $i > 0$ or $i = 0$), with $\hat{x}_n(s, x)$ playing the role of x , and deduce that

$$\begin{aligned}
&|u'_{1,a_n^{\alpha_i}}(s, x') - u'_{1,a_n^{\alpha_i}}(s, \hat{x}_n(s, x))| \\
&\leq 2^{-82} a_n^{-\varepsilon_0 - \frac{3\varepsilon_1}{2}} (|x - x'| + \sqrt{a_n})^{1-\varepsilon_0} \left[a_n^{-3\beta_i/2} (|x' - x| + \sqrt{a_n})^{2\gamma} \right. \\
&\quad \left. + 1 + a_n^{\beta_i(\gamma - \frac{3}{2})} (|x - x'| + \sqrt{a_n})^\gamma \right].
\end{aligned}$$

Use the fact that $\beta_i + 5\varepsilon_1 \leq 1/2$ (recall (3.11)) to infer $|x - x'| + \sqrt{a_n} \leq 6a_n^{\beta_i + 5\varepsilon_1} \leq a_n^{\beta_i}$ (by (3.14)) and so bound the above by

$$\begin{aligned}
&2^{-79} a_n^{-\varepsilon_0 - \frac{3\varepsilon_1}{2}} a_n^{(\beta_i + 5\varepsilon_1)(1-\varepsilon_0)} \left[a_n^{\beta_i(2\gamma - \frac{3}{2})} + 1 + a_n^{\beta_i(2\gamma - \frac{3}{2})} \right] \\
&\leq 2^{-79} a_n^{-\varepsilon_0 - \frac{3\varepsilon_1}{2}} a_n^{(\beta_i + 5\varepsilon_1)(1-\varepsilon_0)} 3 \\
&\leq 2^{-77} a_n^{\beta_i + 1},
\end{aligned}$$

provided that $\beta_{i+1} \leq (\beta_i + 5\varepsilon_1)(1 - \varepsilon_0) - \varepsilon_0 - \frac{3\varepsilon_1}{2}$, or equivalently

$$\varepsilon_0 + (\beta_i + 1)\varepsilon_0 \leq ((7/2) - 5\varepsilon_0)\varepsilon_1.$$

This follows easily from (3.10). We have therefore shown that

$$|u'_{1,a_n^{\alpha_i}}(s, x') - u'_{1,a_n^{\alpha_i}}(s, \hat{x}_n(s, x))| \leq 2^{-77} a_n^{\beta_{i+1}},$$

and so both (a) and (b) are now immediate from Lemma 6.5 (b), (c). ■

Lemma 6.7 *If $i \in \{0, \dots, L\}$, $0 \leq s < U_{M,n}$, $x \in J_{n,i}(s)$, and $|x - x'| \leq 4\sqrt{a_n}$, then*

$$|u_{2,a_n^{\alpha_i}}(s, x') - u_{2,a_n^{\alpha_i}}(s, x'')| \leq 2^{-75} a_n^{\beta_{i+1}} (|x' - x''| \vee a_n^{\gamma-2\beta_i(1-\gamma)-\varepsilon_1}) \text{ whenever } |x' - x''| \leq \bar{l}_n(\beta_i).$$

Proof. Assume (i, n, s, x, x') are as above and set $\varepsilon = 5\sqrt{a_n} \leq 2^{-M}$, by (3.14). Then

$$|x' - \hat{x}_n(s, x)| \leq |x' - x| + \sqrt{a_n} \leq \varepsilon, \quad |x'| \leq |x| + 1 \leq K_0 + 1, \quad |u(s, \hat{x}_n(s, x))| \leq a_n = a_n \wedge (\sqrt{a_n}\varepsilon),$$

and the definition of $(s, x) \in J_{n,i}$ implies that for $i > 0$,

$$|u'_{1,a_n}(s, \hat{x}_n(s, x))| \leq a_n^{\beta_i}/4 \leq a_n^{\beta_i}.$$

Let

$$Q(n, \varepsilon_0, \beta_i, r) = a_n^{-\varepsilon_0} r^{\frac{1-\varepsilon_0}{2}} \left[(\sqrt{a_n} \vee r)^{2\gamma} + a_n^{\beta_i\gamma} (\sqrt{a_n} \vee r)^\gamma \right].$$

Assume $|x' - x''| \leq \bar{l}_n(\beta_i) \leq 2^{-M}$, the last by (3.14). The condition $s < U_{M,n} \leq U_{M,n,\beta_i}^{(2)}$ and the definition of $U^{(2)}$, with (x', x'') playing the role of (x, x') , ensures that

$$\begin{aligned} |u_{2,a_n^{\alpha_i}}(s, x'') - u_{2,a_n^{\alpha_i}}(s, x')| &\leq 2^{-87} a_n^{-\varepsilon_0} \left[|x'' - x'|^{\frac{1-\varepsilon_0}{2}} \left[((5\sqrt{a_n}) \vee |x'' - x'|)^{2\gamma} \right. \right. \\ &\quad \left. \left. + a_n^{\beta_i\gamma} ((5\sqrt{a_n}) \vee |x'' - x'|)^\gamma \right] + |x'' - x'|^{1-\varepsilon_0} a_n^{\beta_i + \frac{\varepsilon_1}{4}} \right] \\ (6.20) \quad &\leq 2^{-82} [Q(n, \varepsilon_0, \beta_i, |x'' - x'|) + |x'' - x'|^{1-\varepsilon_0} a_n^{\beta_i + \frac{\varepsilon_1}{4} - \varepsilon_0}]. \end{aligned}$$

We first show that

$$(6.21) \quad Q(n, \varepsilon_0, \beta_i, r) \leq 2a_n^{\beta_{i+1}} (r \vee a_n^{\gamma-2\beta_i(1-\gamma)-\varepsilon_1}) \text{ for } 0 \leq r \leq \bar{l}_n(\beta_i).$$

Case 1. $\sqrt{a_n} \leq r \leq \bar{l}_n(\beta_i)$.

$$Q(n, \varepsilon_0, \beta_i, r) = a_n^{-\varepsilon_0} \left[r^{2\gamma + \frac{1}{2} - \frac{\varepsilon_0}{2}} + a_n^{\beta_i\gamma} r^{\gamma + \frac{1}{2} - \frac{\varepsilon_0}{2}} \right],$$

and so (6.21), will hold if

$$(6.22) \quad r^{2\gamma - \frac{1}{2} - \frac{\varepsilon_0}{2}} \leq a_n^{\beta_{i+1} + \varepsilon_0},$$

and

$$(6.23) \quad a_n^{\beta_i\gamma} r^{\gamma - \frac{1}{2} - \frac{\varepsilon_0}{2}} \leq a_n^{\beta_{i+1} + \varepsilon_0}.$$

(3.10) implies $2\gamma - \frac{1}{2} - \frac{\varepsilon_0}{2} > 1$ and so (6.22) would follow from

$$r \leq a_n^{\beta_{i+1} + \varepsilon_0}.$$

Hence, by the upper bound on r in this case, it suffices to show that $a_n^{\beta_i + 5\varepsilon_1} \leq a_n^{\beta_i + 2\varepsilon_0}$ and this is immediate from (3.10).

Turning to (6.23), note that

$$\begin{aligned} a_n^{\beta_i \gamma} r^{\gamma - \frac{1}{2} - \frac{\varepsilon_0}{2}} a_n^{-\beta_{i+1} - \varepsilon_0} &\leq a_n^{\beta_i \gamma + (\beta_i + 5\varepsilon_1)(\gamma - \frac{1}{2} - \frac{\varepsilon_0}{2}) - \beta_{i+1} - \varepsilon_0} \\ &\leq a_n^{\beta_i(2\gamma - \frac{3}{2} - \frac{\varepsilon_0}{2}) + 5\varepsilon_1(\gamma - \frac{1}{2} - \frac{\varepsilon_0}{2}) - 2\varepsilon_0} \\ &\leq a_n^{5\varepsilon_1(\gamma - \frac{1}{2} - \frac{\varepsilon_0}{2}) - 2\varepsilon_0} \leq 1, \end{aligned}$$

where in the last two inequalities we are using (3.10). This proves (6.23) and hence completes the derivation of (6.21) in this case.

Case 2. $a_n^{\gamma - 2\beta_i(1-\gamma) - \varepsilon_1} \leq r < \sqrt{a_n}$.

Then

$$Q(n, \varepsilon_0, \beta_i, r) = a_n^{-\varepsilon_0} r^{\frac{1-\varepsilon_0}{2}} \left[a_n^\gamma + a_n^{\gamma(\beta_i + \frac{1}{2})} \right] \leq 2a_n^{-\varepsilon_0} r^{\frac{1-\varepsilon_0}{2}} a_n^{\gamma(\beta_i + \frac{1}{2})},$$

and so (6.21) will hold if

$$(6.24) \quad r^{\frac{1+\varepsilon_0}{2}} \geq a_n^{-\beta_{i+1} - \varepsilon_0 + \gamma(\beta_i + \frac{1}{2})} = a_n^{-\beta_i(1-\gamma) + \frac{\gamma}{2} - 2\varepsilon_0}.$$

Our lower bound on r implies that

$$r^{\frac{1+\varepsilon_0}{2}} \geq a_n^{(\gamma - 2\beta_i(1-\gamma) - \varepsilon_1)(1+\varepsilon_0)/2} \geq a_n^{-\beta_i(1-\gamma) + \frac{\gamma}{2} + \frac{\gamma\varepsilon_0}{2} - \frac{\varepsilon_1}{2}}$$

which implies (6.24) by (3.10).

Case 3. $r < a_n^{\gamma - 2\beta_i(1-\gamma) - \varepsilon_1}$.

This case follows from Case 2 and the monotonicity of $Q(n, \varepsilon_0, \beta_i, r)$ in r . Strictly speaking we also need the fact that Case 2 is non-empty as was done in Lemma 3.6.

This completes the proof of (6.21). Consider next the second term in (6.20). If $r \geq a_n$, then

$$\begin{aligned} r^{1-\varepsilon_0} a_n^{\beta_i + \frac{\varepsilon_1}{4} - \varepsilon_0} (a_n^{\beta_{i+1}} r)^{-1} &= r^{-\varepsilon_0} a_n^{-2\varepsilon_0 + \frac{\varepsilon_1}{4}} \\ &\leq a_n^{-3\varepsilon_0 + \frac{\varepsilon_1}{4}} < 1 \end{aligned}$$

by (3.10). It follows that

$$(6.25) \quad r^{1-\varepsilon_0} a_n^{\beta_i + \frac{\varepsilon_1}{4} - \varepsilon_0} \leq a_n^{\beta_{i+1}} (r \vee a_n) \leq a_n^{\beta_{i+1}} (r \vee a_n^{\gamma - 2\beta_i(1-\gamma) - \varepsilon_1}),$$

the last inequality being trivial.

Insert (6.21) and (6.25) into (6.20) to derive the desired bound. ■

Lemma 6.8 *If $0 \leq s < U_{M,n}$ and $x \in J_{n,0}(s)$, then*

$$(6.26) \quad |u(s, x) - u(s, x')| \leq (\sqrt{a_n} \vee |x' - x|)^{1-\varepsilon_0} \text{ whenever } |x - x'| \leq 2^{-M},$$

and

$$(6.27) \quad |u(s, x')| \leq 3(\sqrt{a_n})^{1-\varepsilon_0} \text{ whenever } |x' - x| \leq \sqrt{a_n}.$$

Proof. As in (6.14), if $\varepsilon = \sqrt{a_n}$, $(s, x) \in J_{n,0}$ implies

$$(6.28) \quad |u(s, \hat{x}_n(s, x))| \leq a_n \leq \varepsilon, \quad |\hat{x}_n(s, x) - x| \leq \varepsilon, \quad \text{and } |x| \leq K_0.$$

In addition, (3.14) ensures that $\varepsilon \leq 2^{-M}$, and so $s < U_{M,n} \leq U_M^{(4)}$ means that

$$|u(s, x') - u(s, x)| \leq (\sqrt{a_n} \vee |x' - x|)^{1-\varepsilon_0} \quad \text{for all } |x' - x| \leq 2^{-M}.$$

This proves (6.26). Next take $x' = \hat{x}_n(s, x)$ in the above inequality and use (6.28) to obtain for $|x' - x| \leq \sqrt{a_n}$,

$$\begin{aligned} |u(s, x')| &\leq |u(s, x') - u(s, x)| + |u(s, x) - u(s, \hat{x}_n(s, x))| + |u(s, \hat{x}_n(s, x))| \\ &\leq 2\sqrt{a_n}^{1-\varepsilon_0} + a_n \leq 3\sqrt{a_n}^{1-\varepsilon_0}. \end{aligned}$$

This proves (6.27). ■

Proof of Proposition 3.3 The compactness is elementary and left for the reader—note here that continuity allows us to replace the closed intervals on which the inequalities defining $\tilde{J}_{n,i}(s)$ hold with open intervals.

The inclusions $J_{n,i}(s) \subset \tilde{J}_{n,i}(s)$ for $0 \leq s < U_{M,n}$ are immediate from Lemmas 6.6, 6.7 and 6.8. ■

This finishes the proof of Proposition 3.3 except for the proof of Proposition 5.14 which is the objective of the next section.

7 Proof of Proposition 5.14

We now continue to assume $b \equiv 0$ and give the proof of Proposition 5.14. Assume first $t' \geq t$ and use (3.5) to write

(7.1)

$$\begin{aligned} &|u_{2,\delta}(t', x') - u_{2,\delta}(t, x)| \\ &\leq \left| \int_{(t-\delta)^+}^{t \wedge (t'-\delta)^+} p_{t-s}(y-x) D(s, y) W(ds, dy) \right| + \left| \int_{(t'-\delta)^+ \vee t}^{t'} p_{t'-s}(y-x') D(s, y) W(ds, dy) \right| \\ &\quad + 1(t' - t < \delta) \left| \int_{(t'-\delta)^+}^t (p_{t'-s}(y-x') - p_{t-s}(y-x)) D(s, y) W(ds, dy) \right|. \end{aligned}$$

This decomposition and (3.2) suggests we define the following square functions for $\delta \in (0, 1]$ and $\eta_0 \in (0, 1/2)$ (noting also that $(t' - \delta)^+ \vee t \geq t' - (\delta \wedge (t' - t))$):

$$\begin{aligned}
\hat{Q}_{T,1,\delta}(t, t', x) &= \int_{(t-\delta)^+}^{t \wedge (t'-\delta)^+} \int p_{t-s}(y-x)^2 e^{2R_1|y|} |u(s, y)|^{2\gamma} dy ds, \\
\hat{Q}_{T,2,\delta}(t, t', x') &= \int_{t'-(\delta \wedge (t'-t))}^{t'} \int p_{t'-s}(y-x')^2 e^{2R_1|y|} |u(s, y)|^{2\gamma} dy ds, \\
\hat{Q}_{S,1,\delta,\eta_0}(t, x, t', x') &= 1(t'-t < \delta) \int_{(t'-\delta)^+}^t \int 1(|y-x| > (t-s)^{1/2-\eta_0} \vee (2|x-x'|)) \\
&\quad \times (p_{t'-s}(y-x') - p_{t-s}(y-x))^2 e^{2R_1|y|} |u(s, y)|^{2\gamma} dy ds, \\
\hat{Q}_{S,2,\delta,\eta_0}(t, x, t', x') &= 1(t'-t < \delta) \int_{(t'-\delta)^+}^t \int 1(|y-x| \leq (t-s)^{1/2-\eta_0} \vee (2|x-x'|)) \\
&\quad \times (p_{t'-s}(y-x') - p_{t-s}(y-x))^2 e^{2R_1|y|} |u(s, y)|^{2\gamma} dy ds.
\end{aligned}$$

Lemma 7.1 For any $K \in \mathbb{N}^{\geq K_1}$ and $R > 2$ there is a $c_{7.1}(K, R) > 0$ and an $N_{7.1} = N_{7.1}(K, \omega) \in \mathbb{N}$ a.s. such that for all $\eta_0, \eta_1 \in (1/R, 1/2)$, $\delta \in (0, 1]$, $N, n \in \mathbb{N}$, $\beta \in [0, 1/2]$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, on

$$(7.2) \quad \{\omega : (t, x) \in Z(N, n, K, \beta), N \geq N_{7.1}\},$$

$$\hat{Q}_{S,1,\delta,\eta_0}(t, x, t', x') \leq c_{7.1}(K, R) 2^{4N_{7.1}} [d((t, x), (t', x')) \wedge \sqrt{\delta}]^{2-\eta_1} \delta^{3/2} \quad \text{for all } t \leq t' \text{ and } x' \in \mathbb{R}.$$

Proof. The proof is quite similar to that of Lemma 5.4. We let $d = d((t, x), (t', x'))$ and $N_{7.1} = N_1(0, 3/4, K)$, where N_1 is as in (P_0) . Recall here from Remark 5.3 that for $m = 0$, N_1 depends only on (ξ, K) and we take $\xi = 3/4$. We may assume $t' - t < \delta$. Then for ω as in (7.2) and $t \leq t'$, Lemma 5.2, with $m = 0$, implies

$$\begin{aligned}
&\hat{Q}_{S,1,\delta,\eta_0}(t, x, t', x') \\
&\leq C_{5.2}(\omega) \int_{(t'-\delta)^+}^t \int 1(|y-x| > (t-s)^{1/2-\eta_0} \vee (2|x-x'|)) (p_{t'-s}(y-x') - p_{t-s}(y-x))^2 \\
&\quad \times e^{2(|y-x|+R_1|y|)} (2^{-N} \vee (\sqrt{t-s} + |y-x|))^{3\gamma/2} dy ds \\
&\leq C_{5.2}(\omega) \int_{(t'-\delta)^+}^t \int 1(|y-x| > (t-s)^{1/2-\eta_0} \vee (2|x-x'|)) (p_{t'-s}(y-x') - p_{t-s}(y-x))^2 \\
&\quad e^{2R_1K} e^{2(R_1+1)|y-x|} [1 + |y-x|]^{3\gamma/2} dy ds \\
&\leq c_1(K, R) C_{5.2}(\omega) \int_{t'-\delta}^t (t-s)^{-1/2} \exp\left(-\frac{\eta_1(t-s)^{-2\eta_0}}{33}\right) \left[1 \wedge \frac{d^2}{t-s}\right]^{1-\frac{\eta_1}{2}} ds,
\end{aligned}$$

where we have used Lemma 4.3(b) in the last line. Now use

$$\exp\left(-\frac{\eta_1(t-s)^{-2\eta_0}}{33}\right) \leq \exp\left(-\frac{\eta_1(t-t)^{-2\eta_0}}{66}\right) + \exp\left(-\frac{\eta_1(t-s)^{-2\eta_0}}{66}\right)$$

to bound the above by

$$\begin{aligned}
& c_1(K, R)C_{5.2}(\omega) \left[\exp\left(-\frac{\eta_1(t' - t)^{-2\eta_0}}{66}\right) \int_{t'-\delta}^t (t-s)^{-1/2} \left[1 \wedge \frac{d^2}{t-s}\right]^{1-\frac{\eta_1}{2}} ds \right. \\
& \quad \left. + \int_{t'-\delta}^t (t-s)^{-1/2} \exp\left(-\frac{\eta_1(t-s)^{-2\eta_0}}{66}\right) \left[1 \wedge \frac{d^2}{t-s}\right]^{1-\frac{\eta_1}{2}} ds \right] \\
& \leq c_2(K, R)C_{5.2}(\omega) \left[\exp\left(-\frac{\eta_1(t' - t)^{-2\eta_0}}{66}\right) (d^2 \wedge \delta)^{1/2} \right. \\
& \quad \left. + \int_{t'-\delta}^t (t-s)^{3/2} \left[1 \wedge \frac{d^2}{t-s}\right]^{1-\frac{\eta_1}{2}} ds \right] \quad (\text{use (4.2)}) \\
& \leq c_3(K, R)C_{5.2}(\omega) (d^2 \wedge \delta)^{1-\frac{\eta_1}{2}} \delta^{3/2} \quad (\text{use (4.1) and } t' - t \leq d^2 \wedge \delta).
\end{aligned}$$

As we may take $\varepsilon_0 = 0$ in the formula for $C_{5.2}(\omega)$ (by Remark 5.3), the result follows. \blacksquare

Lemma 7.2 *Let $0 \leq m \leq \bar{m} + 1$ and assume (P_m) (in (5.5)). For any $K \in \mathbb{N}^{\geq K_1}$, $R > 2$, $n \in \mathbb{N}$, $\varepsilon_0 \in (0, 1)$, and $\beta \in [0, 1/2]$ there is a $c_{7.2}(K)$ and $N_{7.2} = N_{7.2}(m, n, R, \varepsilon_0, K, \beta)(\omega) \in \mathbb{N}$ a.s. such that for any $\eta_1 \in (R^{-1}, 1/2)$, $\eta_0 \in (0, \eta_1/24)$, $\delta \in [a_n, 1]$, $N \in \mathbb{N}$, and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, on*

$$(7.3) \quad \{\omega : (t, x) \in Z(N, n, K, \beta), N \geq N_{7.2}\},$$

$$\begin{aligned}
& \hat{Q}_{S,2,\delta,\eta_0}(t, x, t', x') \\
& \leq c_{7.2}(K) [a_n^{-2\varepsilon_0} + 2^{4N_{7.2}}] \left[(d \wedge \sqrt{\delta})^{1-\frac{\eta_1}{2}} \bar{d}_N^{2\gamma} [\bar{d}_{n,N}^{2\gamma(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma}] \right. \\
& \quad \left. + (d \wedge \sqrt{\delta})^{2-\eta_1} [\delta^{\gamma\tilde{\gamma}_m-\frac{1}{2}} + a_n^{2\beta\gamma}\delta^{\gamma-\frac{1}{2}}] \right] \\
& \quad \text{for all } t \leq t' \leq K, |x'| \leq K + 1.
\end{aligned}$$

Here $d = d((t, x), (t', x'))$, $\bar{d}_N = d \vee 2^{-N}$ and $\bar{d}_{n,N} = \sqrt{a_n} \vee \bar{d}_N$. Moreover $N_{7.2}$ is stochastically bounded uniformly in (n, β) .

Proof. Set $\xi = 1 - (24R)^{-1}$ and $N_{7.2}(m, n, R, \varepsilon_0, K, \beta) = N_1(m, n, \xi, \varepsilon_0, K, \beta)$, which is clearly stochastically bounded uniformly in (n, β) by (P_m) . We may assume $t' - t \leq \delta$. For ω as in (7.3), $t \leq t' \leq K$ and $|x'| \leq K + 1$, Lemma 5.2 implies

$$\begin{aligned}
& \hat{Q}_{S,2,\delta,\eta_0}(t, x, t', x') \\
& \leq C_{5.2}(\omega)c_1(K) \int_{(t'-\delta)^+}^t \left[\int (p_{t'-s}(y-x') - p_{t-s}(y-x))^2 dy \right] e^{2R_1K} e^{2(R_1+1)4(K+1)} \\
& \quad \times ((2^{-N} \vee |x' - x|) + (t-s)^{\frac{1}{2}-\eta_0})^{2\gamma\xi} \left[((\sqrt{a_n} \vee 2^{-N} \vee |x' - x|) \right. \\
& \quad \quad \left. + (t-s)^{\frac{1}{2}-\eta_0})^{2\gamma(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma} \right] ds \\
& \leq C_{5.2}(\omega)c_2(K) \int_{(t'-\delta)^+}^t (t-s)^{-1/2} \left[1 \wedge \frac{d^2}{t-s}\right] \left((2^{-N} \vee d^{1-2\eta_0}) + (t-s)^{\frac{1}{2}-\eta_0} \right)^{2\gamma\xi} \\
& \quad \times \left[((\sqrt{a_n} \vee 2^{-N} \vee d^{1-2\eta_0}) + (t-s)^{\frac{1}{2}-\eta_0})^{2\gamma(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma} \right] ds.
\end{aligned}$$

(7.4)

We have used Lemma 4.3(a) in the last line. Below we will implicitly use the conditions on η_0, η_1, R and γ to see that

$$(1 - 2\eta_0)\gamma\xi > \left(\frac{23}{24}\right)\left(\frac{3}{4}\right)\left(\frac{47}{48}\right) > \frac{1}{2},$$

and also use the conditions on t, x, t', x' which imply $d \leq c(K)$ (the latter was also used in (7.4)). By considering separately the cases

$$(t-s)^{\frac{1}{2}-\eta_0} < 2^{-N} \vee d^{1-2\eta_0}, \quad (t-s)^{\frac{1}{2}-\eta_0} \geq \sqrt{a_n} \vee 2^{-N} \vee d^{1-2\eta_0}$$

$$\text{and } 2^{-N} \vee d^{1-2\eta_0} \leq (t-s)^{\frac{1}{2}-\eta_0} < \sqrt{a_n} \vee 2^{-N} \vee d^{1-2\eta_0}$$

(the latter case implies $\sqrt{a_n} > 2^{-N} \vee d^{1-2\eta_0}$), and then using Lemma 4.1 we may bound (7.4) by

$$\begin{aligned} & C_{5.2}(\omega)c_3(K) \left\{ \int_{(t'-\delta)^+}^t (t-s)^{-1/2} \left[1 \wedge \frac{d^2}{t-s} \right] ds (2^{-N} \vee d^{1-2\eta_0})^{2\gamma\xi} \right. \\ & \quad \times \left[(\sqrt{a_n} \vee 2^{-N} \vee d^{1-2\eta_0})^{2\gamma(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma} \right] \\ & \quad + \int_{(t'-\delta)^+}^t \left[(t-s)^{-\frac{1}{2}+(1-2\eta_0)\gamma(\tilde{\gamma}_m-1+\xi)} + a_n^{2\beta\gamma} (t-s)^{-\frac{1}{2}+(1-2\eta_0)\gamma\xi} \right] \left[1 \wedge \frac{d^2}{t-s} \right] ds \\ & \quad \left. + \int_{(t'-\delta)^+}^t (t-s)^{(1-2\eta_0)\gamma\xi-\frac{1}{2}} \left[1 \wedge \frac{d^2}{t-s} \right] ds [a_n^{\gamma(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma}] \right\} \\ & \leq C_{5.2}(\omega)c_4(K) \left\{ (d^2 \wedge \delta)^{1/2} (2^{-N} \vee d^{1-2\eta_0})^{2\gamma\xi} \right. \\ & \quad \times \left[(\sqrt{a_n} \vee 2^{-N} \vee d^{1-2\eta_0})^{2\gamma(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma} \right] \\ & \quad + (d^2 \wedge \delta) \left[\delta^{(1-2\eta_0)\gamma(\tilde{\gamma}_m-1+\xi)-\frac{1}{2}} + a_n^{2\beta\gamma} \delta^{(1-2\eta_0)\gamma\xi-\frac{1}{2}} \right] \\ & \quad \left. + (d^2 \wedge \delta) \delta^{(1-2\eta_0)\gamma\xi-\frac{1}{2}} [a_n^{\gamma(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma}] \right\} \end{aligned}$$

(by (4.2) and (4.1), respectively).

The last term is less than the middle term because $\delta \in [a_n, 1]$. Therefore $\hat{Q}_{S,2,\delta,\eta_0}(t, x, t', x')$ is at most

$$(7.5) \quad \begin{aligned} & C_{5.2}(\omega)c_5(K) \left\{ (d \wedge \sqrt{\delta})^{1-\frac{\eta_1}{2}} (2^{-N} \vee d)^{(1-2\eta_0)2\gamma\xi+\frac{\eta_1}{4}} \right. \\ & \quad \times \left[(\sqrt{a_n} \vee 2^{-N} \vee d)^{(1-2\eta_0)2\gamma(\tilde{\gamma}_m-1)+\frac{\eta_1}{4}} + a_n^{2\beta\gamma} (\sqrt{a_n} \vee 2^{-N} \vee d)^{\frac{\eta_1}{4}} \right] \\ & \quad \left. + (d^2 \wedge \delta)^{1-\frac{\eta_1}{2}} \left[\delta^{(1-2\eta_0)\gamma(\tilde{\gamma}_m-1+\xi)-\frac{1}{2}+\frac{\eta_1}{2}} + a_n^{2\beta\gamma} \delta^{(1-2\eta_0)\gamma\xi-\frac{1}{2}+\frac{\eta_1}{2}} \right] \right\}. \end{aligned}$$

Our conditions on η_0, η_1 , and R imply

$$(1 - 2\eta_0)2\gamma\xi + \frac{\eta_1}{4} \geq 2\gamma, \quad (1 - 2\eta_0)2\gamma(\tilde{\gamma}_m - 1) + \frac{\eta_1}{4} \geq 2\gamma(\tilde{\gamma}_m - 1),$$

$$(1 - 2\eta_0)\gamma(\tilde{\gamma}_m - 1 + \xi) + \frac{\eta_1}{2} \geq \gamma\tilde{\gamma}_m, \quad \text{and } (1 - 2\eta_0)\gamma\xi + \frac{\eta_1}{2} \geq \gamma.$$

Finally, insert the above bounds into (7.5) to derive the required bound on $\hat{Q}_{S,2,\delta,\eta_0}(t, x, t', x')$. \blacksquare

Lemma 7.3 *Let $0 \leq m \leq \bar{m} + 1$ and assume (P_m) . For any $K \in \mathbb{N}^{\geq K_1}$, $R > 2$, $n \in \mathbb{N}$, $\varepsilon_0 \in (0, 1)$, and $\beta \in [0, 1/2]$ there is a $c_{7.3}(K)$ and $N_{7.3} = N_{7.3}(m, n, R, \varepsilon_0, K, \beta)(\omega) \in \mathbb{N}$ a.s. such that for any $\eta_1 \in (R^{-1}, 1/2)$, $\delta \in [a_n, 1]$, $N \in \mathbb{N}$, and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,*

(7.6) *on $\{\omega : (t, x) \in Z(N, n, K, \beta), N \geq N_{7.3}\}$, and for all $t \leq t' \leq T_K$, $|x'| \leq K + 1$,*

$$\begin{aligned} & \hat{Q}_{T,1,\delta}(t, t', x) \\ & \leq c_{7.3}(K)[a_n^{-2\varepsilon_0} + 2^{4N_{7.3}}] \left[(\sqrt{t'} - t \wedge \sqrt{\delta})^{1-\frac{\eta_1}{2}} \bar{d}_{n,N}^{2\gamma} [\bar{d}_{n,N}^{2\gamma(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma}] \right. \\ & \quad \left. + (\sqrt{t'} - t \wedge \sqrt{\delta})^{2-\eta_1} [\delta^{\gamma\tilde{\gamma}_m-\frac{1}{2}} + a_n^{2\beta\gamma}\delta^{\gamma-\frac{1}{2}}] \right], \end{aligned}$$

and

$$\begin{aligned} & \hat{Q}_{T,2,\delta}(t, t', x') \\ & \leq c_{7.3}(K)[a_n^{-2\varepsilon_0} + 2^{4N_{7.3}}] (\sqrt{t'} - t \wedge \sqrt{\delta})^{1-\frac{\eta_1}{2}} \bar{d}_{n,N}^{2\gamma} \left[\bar{d}_{n,N}^{2\gamma(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma} \right]. \end{aligned}$$

Here d , \bar{d}_N and $\bar{d}_{n,N}$ are as in Lemma 7.2. Moreover, $N_{7.3}$ is stochastically bounded uniformly in (n, β) .

Proof. Set $\xi = 1 - (4R)^{-1}$ and $N_{7.3}(m, n, R, \varepsilon_0, K, \beta) = N_1(m, n, \xi, \varepsilon_0, K, \beta)$, which is clearly stochastically bounded uniformly in (n, β) by (P_m) . For ω , t , x , t' and x' as in (7.6), Lemma 5.2 gives

$$\begin{aligned} & \hat{Q}_{T,2,\delta}(t, t', x') \\ & \leq C_{5.2}(\omega) c_1 \int_{t'-(\delta \wedge (t'-t))}^{t'} \int p_{t'-s}(y - x')^2 e^{2R_1 K + (2(R_1+1))|y-x'| + 2(R_1+1)(2K+1)} \\ & \quad \times [2^{-N} \vee |x - x'| + \sqrt{t' - s} + |y - x'|]^{2\gamma\xi} \\ & \quad \times \left\{ [\sqrt{a_n} \vee 2^{-N} \vee |x - x'| + \sqrt{t' - s} + |y - x'|]^{2\gamma(\tilde{\gamma}_m-1)} + a_n^{2\beta\gamma} \right\} dy ds \\ & \leq C_{5.2}(\omega) c_2(K) \int_{t'-(\delta \wedge (t'-t))}^{t'} (t' - s)^{-1/2} [(2^{-N} \vee |x - x'|)^{2\gamma\xi} + (t' - s)^{\gamma\xi}] \\ (7.7) \quad & \times \left\{ ((\sqrt{a_n} \vee 2^{-N} \vee |x - x'|)^{2\gamma(\tilde{\gamma}_m-1)} + (t' - s)^{\gamma(\tilde{\gamma}_m-1)}) + a_n^{2\beta\gamma} \right\} ds. \end{aligned}$$

For $t \leq s \leq t'$ and $c, p \geq 0$,

$$(c \vee |x - x'|)^p + (t' - s)^{p/2} \leq 2(c \vee d)^p.$$

Use this with $c = 2^{-N}$ or $\sqrt{a_n} \vee 2^{-N}$ to bound (7.7) by

$$\begin{aligned} & C_{5.2}(\omega)c_3(K)((t' - t) \wedge \delta)^{\frac{1}{2}}[2^{-N} \vee d]^{2\gamma\xi} \left\{ (\sqrt{a_n} \vee 2^{-N} \vee d)^{2\gamma(\tilde{\gamma}_m - 1)} + a_n^{2\beta\gamma} \right\} \\ & \leq C_{5.2}(\omega)c_3(K)(\sqrt{t' - t} \wedge \sqrt{\delta})^{1 - \frac{\eta_1}{2}} \bar{d}_N^{2\gamma\xi + \frac{\eta_1}{2}} \left\{ \bar{d}_{n,N}^{2\gamma(\tilde{\gamma}_m - 1)} + a_n^{2\beta\gamma} \right\}. \end{aligned}$$

The conditions on η_1 and definition of ξ imply $2\gamma\xi + \frac{\eta_1}{2} \geq 2\gamma$, and so the bound on $\hat{Q}_{T,2,\delta}$ is established.

Turning to $\hat{Q}_{T,1,\delta}$, we may assume $t' > \delta$, or else $\hat{Q}_{T,1,\delta} = 0$. Argue as in the derivation of (7.7) to see that for ω , t , t' and x as in (7.6),

$$(7.8) \quad \begin{aligned} \hat{Q}_{T,1,\delta}(t, t', x) & \leq C_{5.2}(\omega)c_4(K) \int_{(t-\delta)^+}^{t \wedge (t' - \delta)} (t - s)^{-1/2} [(2^{-2N\gamma\xi} + (t - s)^{\gamma\xi}] \\ & \quad \times \left\{ (\sqrt{a_n} \vee 2^{-N})^{2\gamma(\tilde{\gamma}_m - 1)} + (t - s)^{\gamma(\tilde{\gamma}_m - 1)} + a_n^{2\beta\gamma} \right\} ds. \end{aligned}$$

Elementary calculations give

$$(7.9) \quad \int_{t-\delta}^{t \wedge (t' - \delta)} (t - s)^{-1/2} ds \leq 2(\sqrt{t' - t} \wedge \sqrt{\delta}),$$

and for $p \geq 0$,

$$(7.10) \quad \int_{t-\delta}^{t \wedge (t' - \delta)} (t - s)^p ds \leq \delta^p ((t' - t) \wedge \delta).$$

For the integral in (7.8) consider separately the cases (i) $\sqrt{t - s} < 2^{-N}$, (ii) $\sqrt{t - s} \geq \sqrt{a_n} \vee 2^{-N}$, and (iii) $2^{-N} \leq \sqrt{t - s} < \sqrt{a_n} \vee 2^{-N}$, the latter implying $\sqrt{a_n} \vee 2^{-N} = \sqrt{a_n}$, to bound $\hat{Q}_{T,1,\delta}(t, t', x)$ by

$$\begin{aligned} & C_{5.2}(\omega)c_4(K) \left\{ \int_{(t-\delta)^+}^{t \wedge (t' - \delta)} (t - s)^{-1/2} ds 2^{-2N\gamma\xi} \left[(\sqrt{a_n} \vee 2^{-N})^{2\gamma(\tilde{\gamma}_m - 1)} + a_n^{2\beta\gamma} \right] \right. \\ & \quad + \int_{(t-\delta)^+}^{t \wedge (t' - \delta)} (t - s)^{\gamma(\tilde{\gamma}_m + \xi - 1) - \frac{1}{2}} + a_n^{2\beta\gamma} (t - s)^{\gamma\xi - \frac{1}{2}} ds \\ & \quad \left. + \int_{(t-\delta)^+}^{t \wedge (t' - \delta)} (t - s)^{\gamma\xi - \frac{1}{2}} ds \left[a_n^{\gamma(\tilde{\gamma}_m - 1)} + a_n^{2\beta\gamma} \right] \right\} \\ & \leq C_{5.2}(\omega)c_5(K) \left\{ (\sqrt{\delta} \wedge \sqrt{t' - t}) \bar{d}_N^{-2\gamma\xi} \left[\bar{d}_{n,N}^{2\gamma(\tilde{\gamma}_m - 1)} + a_n^{2\beta\gamma} \right] \right. \\ & \quad + (\delta \wedge (t' - t)) \left[\delta^{\gamma(\tilde{\gamma}_m + \xi - 1) - \frac{1}{2}} + a_n^{2\beta\gamma} \delta^{\gamma\xi - \frac{1}{2}} \right] \\ & \quad \left. + (\delta \wedge (t' - t)) \left[\delta^{\gamma\xi - \frac{1}{2}} a_n^{\gamma(\tilde{\gamma}_m - 1)} + \delta^{\gamma\xi - \frac{1}{2}} a_n^{2\beta\gamma} \right] \right\}. \end{aligned}$$

In the last we have used (7.9) and (7.10), and the fact that our choice of ξ implies $\gamma\xi > 1/2$ and hence our choices of p are indeed non-negative when applying (7.10). Since $\delta \geq a_n$, the third term

above is dominated by the second term. Therefore

$$\begin{aligned} \hat{Q}_{T,1,\delta}(t, t', x) &\leq C_{5,2}(\omega) c_6(K) \left\{ (\sqrt{\delta} \wedge \sqrt{t' - t})^{1 - \frac{\eta_1}{2}} \bar{d}_N^{2\gamma\xi + \frac{\eta_1}{2}} \left[\bar{d}_{n,N}^{2\gamma(\tilde{\gamma}_m - 1)} + a_n^{2\beta\gamma} \right] \right. \\ &\quad \left. + (\delta \wedge (t' - t))^{1 - \frac{\eta_1}{2}} \left[\delta^{\gamma(\tilde{\gamma}_m + \xi - 1) - \frac{1}{2} + \frac{\eta_1}{2}} + a_n^{2\beta\gamma} \delta^{\gamma\xi - \frac{1}{2} + \frac{\eta_1}{2}} \right] \right\}. \end{aligned}$$

Our choice of ξ and conditions on η_1 imply that

$$2\gamma\xi + \frac{\eta_1}{2} \geq 2\gamma \text{ and } \gamma(\xi - 1) + \frac{\eta_1}{2} \geq 0,$$

and the required bound on $\hat{Q}_{T,1,\delta}$ follows. ■

The above square function bounds suggest we will need a modified form of Lemma 5.7 to obtain our modulus of continuity for u_{2,a_n^α} . The proof of the following result is almost identical to that of Lemma 5.7 and so is omitted.

Lemma 7.4 *Let c_0, c_1, c_2, k_0 be positive (universal constants), $\eta \in (0, 1/2)$, and $\Delta_i : \mathbb{N} \times (0, 1] \rightarrow \mathbb{R}_+$, $i = 1, 2$ satisfy $\Delta_i(n, 2^{-N+1}) \leq k_0 \Delta_i(n, 2^{-N})$ for all $n, N \in \mathbb{N}$ and $i = 1, 2$. For $n \in \mathbb{N}$ and τ in a set S assume $\{Y_{\tau,n}(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}$ is a real-valued continuous process. Assume for each $(n, \tau) \in \mathbb{N} \times S$, $K \in \mathbb{N}$, and $\beta \in [0, 1/2]$, there is an $N_0(\omega) = N_0(n, \eta, K, \tau, \beta)(\omega) \in \mathbb{N}$ a.s., stochastically bounded uniformly in (n, τ, β) , such that for any $N \in \mathbb{N}$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, if $d = d((t, x), (t', x')) \leq 2^{-N}$, then*

$$(7.11) \quad \begin{aligned} P(|Y_{\tau,n}(t, x) - Y_{\tau,n}(t', x')| > d^{\frac{1}{2}(1-\eta)} \Delta_1(n, 2^{-N}) + d^{1-\eta} \Delta_2(n, 2^{-N}), \\ (t, x) \in Z(N, n, K, \beta), N \geq N_0, t' \leq T_K) \\ \leq c_0 \exp(-c_1 d^{-\eta c_2}). \end{aligned}$$

Then there is an $N'_0(\omega) = N'_0(n, \eta, K, \tau, \beta)(\omega) \in \mathbb{N}$ a.s., also stochastically bounded uniformly in (n, τ, β) , such that for all $N \geq N'_0(\omega)$, $(t, x) \in Z(N, n, K, \beta)(\omega)$, $d = d((t, x), (t', x')) \leq 2^{-N}$, and $t' \leq T_K$,

$$|Y_{\tau,n}(t, x) - Y_{\tau,n}(t', x')| \leq 2^7 k_0^2 \left[d^{\frac{1}{2}(1-\eta)} \Delta_1(n, 2^{-N}) + d^{1-\eta} \Delta_2(n, 2^{-N}) \right].$$

Proof of Proposition 5.14. The proof follows closely that of Proposition 5.8, using Lemma 7.4 in place of Lemma 5.7. Let $R = \frac{25}{\eta_1}$ and choose $\eta_0 \in (\frac{1}{R}, \frac{\eta_1}{24})$. Define $\bar{d}_N = d \vee 2^{-N}$, as usual, and set

$$\hat{Q}_{a_n^\alpha}(t, x, t', x') = \sum_{i=1}^2 \hat{Q}_{S,i,a_n^\alpha,\eta_0}(t, x, t', x') + \hat{Q}_{T,1,a_n^\alpha}(t, t', x) + \hat{Q}_{T,2,a_n^\alpha}(t, t', x').$$

By Lemmas 7.1, 7.2 and 7.3 with $\delta = a_n^\alpha$, for all $K \in \mathbb{N}$ (the restriction $K \geq K_1$ is illusory as these results only strengthen as K increases) there is a constant $c_1(K, \eta_1)$ and $N_2(m, n, \eta_1, \varepsilon_0, K, \beta) \in \mathbb{N}$ a.s., stochastically bounded uniformly in (n, β) , such that for all $N \in \mathbb{N}$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\text{on } \{\omega : (t, x) \in Z(N, n, K + 1, \beta), N \geq N_2\},$$

$$\begin{aligned}
& R_0^\gamma \hat{Q}_{a_n^\alpha}(t, x, t', x')^{1/2} \\
& \leq c_1(K, \eta_1) [a_n^{-\varepsilon_0} + 2^{2N_2}] \left\{ (d \wedge a_n^{\frac{\alpha}{2}})^{\frac{1}{2}(1-\frac{\eta_1}{2})} \bar{d}_N^\gamma \left[(\bar{d}_N \vee a_n^{1/2})^{\gamma(\tilde{\gamma}_m-1)} + a_n^{\beta\gamma} \right] \right. \\
& \quad \left. + (d \wedge a_n^{\frac{\alpha}{2}})^{1-\frac{\eta_1}{2}} \left[a_n^{\frac{\alpha}{2}(\gamma\tilde{\gamma}_m-\frac{1}{2})} + a_n^{\beta\gamma} a_n^{\frac{\alpha}{2}(\gamma-\frac{1}{2})} \right] \right\}
\end{aligned}$$

(7.12)

for all $t \leq t' \leq T_K$, $|x'| \leq K + 2$.

Let $N_3 = \frac{25}{\eta_1} [N_2 + N_4(K, \eta_1)]$, where $N_4(K, \eta_1)$ is chosen large enough so that

$$\begin{aligned}
c_1(K, \eta_1) [a_n^{-\varepsilon_0} + 2^{2N_2}] 2^{-N_3\eta_1/8} & \leq c_1(K, \eta_1) [a_n^{-\varepsilon_0} + 2^{2N_2}] 2^{-3N_2} 2^{-3N_4(K, \eta_1)} \\
& \leq a_n^{-\varepsilon_0} 2^{-104}.
\end{aligned}$$

Let $\Delta_{i, u_2} = 2^{-100} \bar{\Delta}_{i, u_2}$, $i = 1, 2$. Assume $d \leq 2^{-N}$. Use (7.13) in (7.12) to see that for all (t, x) , N , on

$$\{\omega : (t, x) \in Z(N, n, K + 1, \beta), N \geq N_3\} \text{ (which implies } |x'| \leq K + 2),$$

$$\begin{aligned}
R_0^\gamma \hat{Q}_{a_n^\alpha}(t, x, t', x')^{1/2} & \leq (d \wedge a_n^{\frac{\alpha}{2}})^{\frac{1}{2}(1-\frac{3\eta_1}{4})} \Delta_{1, u_2}(m, n, 2^{-N}) / 16 \\
& \quad + (d \wedge a_n^{\frac{\alpha}{2}})^{1-\frac{5\eta_1}{8}} \Delta_{2, u_2}(m, n) / 16 \quad \text{for all } t \leq t' \leq T_K, x' \in \mathbb{R}.
\end{aligned}$$

Combine this with (7.1), (3.2), the definition of $\hat{Q}_{a_n^\alpha}$ and the Dubins-Schwarz theorem to conclude that for $t \leq t'$, $x' \in \mathbb{R}$, and $d((t, x), (t', x')) \leq 2^{-N}$,

$$\begin{aligned}
& P\left(|u_{2, a_n^\alpha}(t, x) - u_{2, a_n^\alpha}(t', x')| \geq d^{\frac{1}{2}(1-\eta_1)} \Delta_{1, u_2}(m, n, 2^{-N}) / 4 + d^{1-\eta_1} \Delta_{2, u_2}(m, n) / 4, \right. \\
& \quad \left. (t, x) \in Z(N, n, K + 1, \beta), N \geq N_3, t' \leq T_K\right) \\
& \leq 3P\left(\sup\{|B(u)| : u \leq [d^{\frac{1}{2}(1-\frac{3\eta_1}{4})} \Delta_{1, u_2}(m, n, 2^{-N}) / 16 + d^{1-\frac{5\eta_1}{8}} \Delta_{2, u_2}(m, n) / 16]^2\} \right. \\
& \quad \left. \geq (d^{\frac{1}{2}(1-\eta_1)} \Delta_{1, u_2}(m, n, 2^{-N}) + d^{1-\eta_1} \Delta_{2, u_2}(m, n)) / 12\right) \\
(7.14) \quad & \leq 3P\left(\sup_{u \leq 1} |B(u)| \geq d^{-\frac{\eta_1}{8}}\right) \leq c_0 \exp\left(-\frac{d^{-\frac{\eta_1}{4}}}{2}\right).
\end{aligned}$$

Here $B(u)$ is a one-dimensional Brownian motion.

If $(t, x) \in Z(N, n, K, \beta)$, $d \leq 2^{-N}$, and $t' \leq t$, then as in the proof of Proposition 5.8, $(t', x') \in Z(N - 1, n, K + 1, \beta)$ and one can interchange the roles of (t, x) and (t', x') and replace N with $N - 1$ in the above to conclude (as in (5.43)),

$$\begin{aligned}
& P(|u_{2, a_n^\alpha}(t, x) - u_{2, a_n^\alpha}(t', x')| \geq d^{\frac{1}{2}(1-\eta_1)} \Delta_{1, u_2}(m, n, 2^{-N}) + d^{1-\eta_1} \Delta_{2, u_2}(m, n), \\
& \quad (t, x) \in Z(N, n, K, \beta), N \geq N_3 + 1) \\
(7.15) \quad & \leq c_0 \exp\left(-\frac{d^{-\frac{\eta_1}{4}}}{2}\right).
\end{aligned}$$

(7.14) and (7.15) allow us to apply Lemma 7.4 with $\tau = \alpha$, $Y_{\tau, n} = u_{2, a_n^\alpha}$, and $k_0 = 4$. The result is then immediate once one recalls that $2^7 k_0^2 \Delta_{i, u_2} = 2^{-89} \bar{\Delta}_{i, u_2}$. \blacksquare

8 Incorporating Drifts

Beginning in Section 3 we assumed that the drift b is zero. Here we point out what additional reasoning is needed to include a drift b satisfying (1.4). If $B(s, y) = b(s, y, X^1(s, y)) - b(s, y, X^2(s, y))$, then (3.1) becomes

$$(8.1) \quad \begin{aligned} u(t, x) &= \int_0^t \int p_{t-s}(y-x) D(s, y) W(ds, dy) + \int_0^t \int p_{t-s}(y-x) B(s, y) dy ds \text{ a.s. for all } (t, x) \\ &\equiv u_D(t, x) + u_B(t, x), \end{aligned}$$

and by (1.4),

$$(8.2) \quad |B(s, y)| \leq B|u(s, y)|.$$

If $u_{1,\delta}, u_{2,\delta}$ and G_δ are defined as at the beginning of Section 3, then as for (3.4), (3.5) and Lemma 3.1, but now using the ordinary Fubini theorem for u_B , we get

$$\begin{aligned} u_{1,\delta}(t, x) &= \int_0^{(t-\delta)^+} \int p_{t-s}(y-x) D(s, y) W(ds, dy) + \int_0^{(t-\delta)^+} \int p_{t-s}(y-x) B(s, y) dy ds \\ &\equiv u_{1,D,\delta}(t, x) + u_{1,B,\delta}(t, x), \\ u_{2,\delta}(t, x) &= \int_{(t-\delta)^+}^t \int p_{t-s}(y-x) D(s, y) W(ds, dy) + \int_{(t-\delta)^+}^t \int p_{t-s}(y-x) B(s, y) dy ds \\ &\equiv u_{2,D,\delta}(t, x) + u_{2,B,\delta}(t, x), \end{aligned}$$

and

$$\begin{aligned} -G'_\delta(s, t, x) &\equiv F_\delta(s, t, x) \\ &= \int_0^{(s-\delta)^+} \int p'_{(t \vee s)-r}(y-x) D(r, y) W(dr, dy) + \int_0^{(s-\delta)^+} \int p'_{(t \vee s)-r}(y-x) B(r, y) dy dr \\ &\equiv F_{D,\delta}(s, t, x) + F_{B,\delta}(s, t, x). \end{aligned}$$

In addition, no changes are required in the verification of (P_0) (including the refinement noted in Remark 5.3) or the proof of Lemma 5.2.

The theorems in Section 5 apply directly to quantities like $u_{i,D,\delta}$ and $F_{D,\delta}$. The corresponding expressions $u_{i,B,\delta}$ and $F_{B,\delta}$ are in fact much easier to handle because we are dealing with a deterministic integral and so regularity properties are easy to read off directly from the bounds in Lemma 5.2. Furthermore, the Lipschitz condition on b effectively sets $\gamma = 1$ for these calculations. To illustrate this, we now prove a simple result which includes both Propositions 5.8 and 5.11 for F_{B,a_n^α} and only requires (P_0) , a consequence of the “crude” modulus Theorem 2.3, already noted above.

Proposition 8.1 For any $\eta_1 \in (0, \frac{1}{2})$ and $K \in \mathbb{N}^{\geq K_1}$ there is an $N_{8.1}(\eta_1, K)(\omega) \in \mathbb{N}$ a.s. such that for all $n \in \mathbb{N}$, $\alpha \in [0, 1]$ and $\beta \in [0, \frac{1}{2}]$, $N \geq N_{8.1}(\eta_1, K)$, $(t, x) \in Z(N, n, K, \beta)$, and $t' \leq T_K$,

$$d = d((t, x), (t', x')) \leq 2^{-N} \text{ and } |s' - s| \leq N^{-1} \text{ imply}$$

$$|F_{B, a_n^\alpha}(s, t, x) - F_{B, a_n^\alpha}(s', t', x')| \leq 2^{-88} \left[|s' - s|^{\frac{1}{2}} 2^{-N(1-\eta_1)} + |s' - s|^{1-\frac{\eta_1}{2}} + d^{1-\eta_1} (1 + a_n^{-3\alpha/4} 2^{-2N}) \right].$$

Proof. Let $\xi = 1 - \frac{\eta_1}{4}$ and assume first that

$$(8.3) \quad N \geq N_1(0, \xi, K + 1) + 1,$$

where N_1 is as in (P_0) and Remark 5.3. Assume

$$(8.4) \quad (t, x) \in Z(N, n, K, \beta), t' \leq T_K, d \leq 2^{-N} \text{ and } |s' - s| \leq N^{-1}.$$

One easily checks that $|t \wedge s - t' \wedge s'| \leq N^{-1}$ and so, by replacing (s, s') with $(t \wedge s, t' \wedge s')$, we may assume that $s \leq t$ and $s' \leq t'$. Define $\bar{s} = s \vee s'$ and $\underline{s} = s \wedge s'$. As before, $(t', x') \in Z(N - 1, n, K + 1, \beta)$, and again, by interchanging (t, x) with (t', x') , we may assume $t \leq t'$ (this is the reason for the $K + 1$ and adding 1 to N_1 in (8.3)). As for (5.8), (8.2) implies

$$\begin{aligned} & |F_{B, a_n^\alpha}(s, t, x) - F_{B, a_n^\alpha}(s', t', x')| \\ & \leq \int_{(\underline{s}-a_n^\alpha)^+}^{(\bar{s}-a_n^\alpha)^+} \int |p'_{t'-r}(y-x')| B|u(r, y)| dy dr + \int_0^{(s-a_n^\alpha)^+} \int |p'_{t'-r}(y-x') - p'_{t'-r}(y-x)| B|u(r, y)| dy dr \\ & \equiv T_1 + T_2. \end{aligned}$$

To bound T_1 , we may assume $\bar{s} \geq a_n^\alpha$. Elementary inequalities using $t' \leq T_K \leq K$, show that for $p \geq 0$,

$$(8.5) \quad \int |p'_{t'-r}(y-x')| |y-x'|^p e^{|y-x'|} dy \leq (t'-r)^{-1} \int |y-x'|^{p+1} e^{|y-x'|} p'_{t'-r}(y-x') dy \leq c_1(K, p)(t'-r)^{\frac{p-1}{2}}.$$

Now apply Lemma 5.2 with $m = 0$ to see that

$$\begin{aligned} T_1 & \leq B \sqrt{C_{5.2}} \int_{(\underline{s}-a_n^\alpha)^+}^{\bar{s}-a_n^\alpha} \int |p'_{t'-r}(y-x')| e^{|y-x'|} e^{|x-x'|} ((\sqrt{t'-r} + |y-x'|) \vee 2^{-N})^\xi dy dr \\ & \leq c_2(K) \sqrt{C_{5.2}} \int_{(\underline{s}-a_n^\alpha)^+}^{\bar{s}-a_n^\alpha} \int |p'_{t'-r}(y-x')| e^{|y-x'|} [(|x-x'| \vee 2^{-N})^\xi + \sqrt{t'-r}^\xi + |y-x'|^\xi] dy dr \\ & \leq c_3(K) \sqrt{C_{5.2}} \int_{(\underline{s}-a_n^\alpha)^+}^{\bar{s}-a_n^\alpha} (t'-r)^{-1/2} [2^{-N\xi} + (t'-r)^{\xi/2}] dr \quad (\text{by (8.5) and } |x-x'| \leq 2^{-N}) \\ & \leq c_4(K) 2^{2N_1(\omega)} [|s' - s|^{1/2} 2^{-N(1-\frac{\eta_1}{4})} + |s' - s|^{1-\frac{\eta_1}{8}}] \\ & \leq c_4(K) 2^{2N_1(\omega)} [2^{-N\eta_1/2} + |s' - s|^{\eta_1/4}] [|s' - s|^{1/2} 2^{-N(1-\eta_1)} + |s - s'|^{1-\frac{\eta_1}{2}}] \\ (8.6) \quad & \leq c_4(K) 2^{2N_1(\omega)} [2^{-N\eta_1/2} + N^{-\eta_1/4}] [|s' - s|^{1/2} 2^{-N(1-\eta_1)} + |s - s'|^{1-\frac{\eta_1}{2}}]. \end{aligned}$$

Recall in the above that we may set $\varepsilon_0 = 0$ in the definition of $C_{5.2}$ (see Remark 5.3).

For T_2 we use both $|u(r, y)| \leq Ke^{|y|}$ for $r \leq t' \leq T_K$ ($K \geq K_1$) and Lemma 5.2 with $m = 0$ to write (we may assume $s > a_n^\alpha$),

$$\begin{aligned}
T_2 &\leq \int_0^{s-a_n^\alpha} \int |p'_{t'-r}(y-x') - p'_{t-r}(y-x)| BK e^{|y|} 1(|y-x| > (t'-r)^{\frac{1}{2}-\frac{\eta_1}{4}} \vee (2|x'-x|)) dy dr \\
&\quad + \int_0^{s-a_n^\alpha} \int |p'_{t'-r}(y-x') - p'_{t-r}(y-x)| B \sqrt{C_{5.2}} e^{|y-x|} [(\sqrt{t-r} + |y-x|) \vee 2^{-N}]^\xi \\
&\quad \quad \quad \times 1(|y-x| \leq (t'-r)^{\frac{1}{2}-\frac{\eta_1}{4}} \vee (2|x'-x|)) dy dr \\
&\leq c_5(K) \sqrt{C_{5.2}} \left[\int_0^{s-a_n^\alpha} \int |p'_{t'-r}(y-x') - p'_{t-r}(y-x)| e^{2|y-x|} \right. \\
&\quad \quad \quad \times 1(|y-x| > (t'-r)^{\frac{1}{2}-\frac{\eta_1}{4}} \vee (2|x'-x|)) e^{-|y-x|} dy dr \\
&\quad \quad \quad \left. + \int_0^{s-a_n^\alpha} \int |p'_{t'-r}(y-x') - p'_{t-r}(y-x)| 1(|y-x| \leq 2(2K+1)) dy ((t'-r)^{\frac{1}{2}-\frac{\eta_1}{4}} \vee |x'-x| \vee 2^{-N})^\xi dr \right] \\
&\leq c_6(K) \sqrt{C_{5.2}} \left[\int_0^{s-a_n^\alpha} \int (p'_{t'-r}(y-x') - p'_{t-r}(y-x))^2 e^{4|y-x|} \right. \\
&\quad \quad \quad \times 1(|y-x| > (t'-r)^{\frac{1}{2}-\frac{\eta_1}{4}} \vee (2|x'-x|)) dy^{1/2} \left(\int e^{-2|y-x|} dy \right)^{1/2} dr \\
&\quad \quad \quad \left. + \int_0^{s-a_n^\alpha} \int (p'_{t'-r}(y-x') - p'_{t-r}(y-x))^2 dy^{1/2} (2(2K+1))^{1/2} [(t-r)^{1/2} \vee 2^{-N}]^{\xi(1-\frac{\eta_1}{2})} dr \right],
\end{aligned}$$

where in the last line for the second term we use $d \leq 2^{-N}$ and

$$\sqrt{t'-r}^{1-\frac{\eta_1}{2}} \leq \sqrt{t'-t}^{1-\frac{\eta_1}{2}} + \sqrt{t-r}^{1-\frac{\eta_1}{2}} \leq d^{1-\frac{\eta_1}{2}} + \sqrt{t-r}^{1-\frac{\eta_1}{2}}.$$

Now apply Lemma 4.4 and conclude

$$\begin{aligned}
T_2 &\leq c_7(K, \eta_1) \sqrt{C_{5.2}} \left\{ \int_0^{s-a_n^\alpha} (t-r)^{-3/4} \exp\left\{ \frac{-\eta_1(t'-r)^{-\eta_1/2}}{128} \right\} \left(1 \wedge \frac{d^2}{t-r}\right)^{\frac{1}{2}-\frac{\eta_1}{4}} dr \right. \\
&\quad \left. + \int_0^{s-a_n^\alpha} (t-r)^{-3/4} \left(1 \wedge \frac{d^2}{t-r}\right)^{1/2} [(t-r)^{1/2} \vee 2^{-N}]^{\xi(1-\frac{\eta_1}{2})} dr \right\} \\
&\leq c_7(K, \eta_1) \sqrt{C_{5.2}} \left\{ \int_0^{s-a_n^\alpha} (t-r)^{-3/4} \exp\left\{ \frac{-\eta_1(t'-r)^{-\eta_1/2}}{128} \right\} \left(1 \wedge \frac{d^2}{t-r}\right)^{\frac{1}{2}-\frac{\eta_1}{4}} dr \right. \\
&\quad \left. + \int_0^{s-a_n^\alpha} (t-r)^{-\frac{3}{4}+\frac{\xi}{2}(1-\frac{\eta_1}{2})} \left(1 \wedge \frac{d^2}{t-r}\right)^{1/2} dr \right. \\
&\quad \left. + \int_0^{s-a_n^\alpha} 1(r \geq t-2^{-2N}) (t-r)^{-\frac{3}{4}} \left(1 \wedge \frac{d^2}{t-r}\right)^{1/2} dr 2^{-N\xi(1-\frac{\eta_1}{2})} \right\} \\
&\leq c_8(K, \eta_1) \sqrt{C_{5.2}} \left\{ d^{1-\frac{\eta_1}{2}} \int_0^{t-d^2} (t-r)^{-\frac{3}{4}-\frac{1}{2}+\frac{\eta_1}{4}} \exp\left\{ \frac{-\eta_1(t-r)^{-\eta_1/2}}{256} \right\} dr \right. \\
&\quad \left. + \exp\left\{ \frac{-\eta_1 d^{-\eta_1}}{256} \right\} \int_{t-d^2}^t (t-r)^{-\frac{3}{4}} dr + d \right. \\
&\quad \left. + 1(a_n^\alpha < 2^{-2N}) d(a_n^\alpha)^{-1/4} 2^{-N\xi(1-\frac{\eta_1}{2})} \right\} \\
&\leq c_9(K, \eta_1) \sqrt{C_{5.2}} \left\{ d^{1-\frac{\eta_1}{2}} + 1(a_n^\alpha < 2^{-2N}) d a_n^{-3\alpha/4} a_n^{\alpha/2} 2^{-N\xi(1-\frac{\eta_1}{2})} \right\} \\
&\leq c_9(K, \eta_1) \sqrt{C_{5.2}} d^{1-\frac{3\eta_1}{4}} \left[1 + a_n^{-\frac{3\alpha}{4}} 2^{-\frac{3N\eta_1}{4}} 2^{-N} 2^{-N(1-\frac{3\eta_1}{4})} \right] \\
(8.7) \quad &\leq c_{10}(K, \eta_1) 2^{2N_1(\omega)} 2^{-N\eta_1/4} d^{1-\eta_1} [1 + a_n^{-\frac{3\alpha}{4}} 2^{-2N}].
\end{aligned}$$

We have used Lemma 4.1 in the above with a bit of algebra to see which case applies, and in the last two lines again used $d \leq 2^{-N}$. (8.6) and (8.7) together show there is an $N_{8.1}(\eta_1, K)(\omega) \in \mathbb{N}$ a.s. such that $N_{8.1}(\eta_1, K) \geq N_1(0, \xi, K+1) + 1$ and, if $N \geq N_{8.1}$, then

$$T_1 + T_2 \leq 2^{-88} \left[|s' - s|^{1/2} 2^{-N(1-\eta_1)} + |s' - s|^{1-\frac{\eta_1}{2}} + d^{1-\eta_1} [1 + a_n^{-\frac{3\alpha}{4}} 2^{-2N}] \right].$$

■

A bit of arithmetic shows that the above bound in the contexts of Propositions 5.8 and 5.11 lead to upper bounds that are bounded by the ones obtained there for increments of F_{D, a_n^α} . For Proposition 5.8 one only needs the first two terms in $\bar{\Delta}_{u'_1}$ and we leave this easy check for the reader. For Proposition 5.11 we may set $(s', t', x') = (t, t, x)$ in the above so that the upper bound becomes (recall $|t-s| \leq 1$ and $\gamma\tilde{\gamma}_m - \frac{3}{2} \leq \frac{1}{2} \leq 1$)

$$\begin{aligned}
&2^{-88} [|t-s|^{1/2} 2^{-N(1-\eta_1)} + |t-s|^{1-\frac{\eta_1}{2}}] \\
&\leq 2^{-88} [2^{-N(1-\eta_1)} + (t-s)^{\frac{1-\eta_1}{2}} (\sqrt{t-s} \vee \sqrt{a_n})^{\gamma\tilde{\gamma}_m - \frac{3}{2}}],
\end{aligned}$$

which is bounded by two of the terms on the right-hand side of the upper bound in Proposition 5.11. Hence we may combine these bounds for F_{B, a_n^α} with those derived in Propositions 5.8 and 5.11 for F_{D, a_n^α} and hence complete the proofs of Propositions 5.8 and 5.11 (and hence also Corollary 5.9) for solutions with Lipschitz drifts.

We omit the analogues of the above for Propositions 5.12 and 5.14 as they are even simpler. The proof of Propositions 5.13 and 5.1 now proceed as before. With Propositions 5.1, 5.13, 5.14, and 5.11, and Corollary 5.9 in hand, the proof of Proposition 3.3 may now be completed for Lipschitz drifts b , exactly as in Section 6. Then verification of the hypotheses of Proposition 2.1 may now be completed for Lipschitz drifts b exactly as in Section 3 and this finishes the proof of Theorem 1.2.

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