

# SELF-INTERSECTION LOCAL TIME OF $(\alpha, d, \beta)$ -SUPERPROCESS

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ABSTRACT. The existence of self-intersection local time (SILT), when the time diagonal is intersected, of the  $(\alpha, d, \beta)$ -superprocess is proved for  $d/2 < \alpha$  and for a renormalized SILT when  $d/(2 + (1 + \beta)^{-1}) < \alpha \leq d/2$ . We also establish Tanaka-like formula for SILT.

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## 1. INTRODUCTION AND STATEMENT OF RESULTS

This paper is devoted to the proof of existence of self-intersection local time (SILT) of  $(\alpha, d, \beta)$ -superprocesses for  $0 < \beta < 1$ . Let us introduce some notation. Let  $\mathcal{B}_b(\mathbb{R}^d)$  (respectively  $\mathcal{C}_b(\mathbb{R}^d)$ ) be the family of all bounded (respectively, bounded continuous) Borel measurable functions on  $\mathbb{R}^d$ , and  $\mathcal{M}_F(\mathbb{R}^d)$  be the set of all finite Borel measures on  $\mathbb{R}^d$ . The integral of a function  $f$  with respect to a measure  $\mu$  is denoted by  $\mu(f)$ . If  $E$  is a metric space we denote by  $D([0, +\infty), E)$  the space of all càdlàg  $E$ -valued paths with the Skorohod topology. We will use  $c$  to denote a positive and finite constant whose value may vary from place to place. A constant of the form  $c(a, b, \dots)$  means that this constant depends on parameters  $a, b, \dots$ .

Let  $(\Omega', \mathcal{F}', \mathcal{F}', P')$  be a filtered probability space where the  $(\alpha, d, \beta)$ -superprocess  $X = \{X_t : t \geq 0\}$  is defined. That is, by  $X$  we mean a  $\mathcal{M}_F(\mathbb{R}^d)$ -valued, time homogeneous, strong Markov process with càdlàg sample paths, such that for any non-negative function  $\varphi \in \mathcal{B}_b(\mathbb{R}^d)$ ,

$$E[\exp(-X_t(\varphi)) | X_0 = \mu] = \exp(-\mu(V_t(\varphi))),$$

where  $\mu \in \mathcal{M}_F(\mathbb{R}^d)$  and  $V_t(\varphi)$  denotes the unique non-negative solution of the following evolution equation

$$v_t = S_t \varphi - \int_0^t S_{t-s} ((v_s)^{1+\beta}) ds, \quad t \geq 0.$$

Here  $\{S_t : t \geq 0\}$  denotes the semigroup corresponding to the fractional Laplacian operator  $\Delta_\alpha$ .

Another way to characterize the  $(\alpha, d, \beta)$ -superprocess  $X$  is by means of the following martingale problem:

$$\begin{cases} \text{For all } \varphi \in D(\Delta_\alpha) \text{ (domain of } \Delta_\alpha) \text{ and } \mu \in \mathcal{M}_F(\mathbb{R}^d), \\ X_0 = \mu, \text{ and } M_t(\varphi) = X_t(\varphi) - X_0(\varphi) - \int_0^t X_s(\Delta_\alpha \varphi) ds, \\ \text{is a } \mathcal{F}_t\text{-martingale.} \end{cases} \quad (1.1)$$

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If  $\beta = 1$  then  $M.(\varphi)$  is a continuous martingale. In this paper we are interested in the case of  $0 < \beta < 1$ , and here  $M_t(\varphi)$  is a purely discontinuous martingale. This martingale can be expressed as

$$M_t(\varphi) = \int_0^t \int_{\mathbb{R}^d} \varphi(x) M(ds, dx), \quad (1.2)$$

where  $M(ds, dx)$  is the martingale measure and the stochastic integral with respect to it is defined in [12] (or in Section II.3 of [10]).

The SILT is heuristically defined by

$$\gamma_X(B) = \int_B \int_{\mathbb{R}^{2d}} \delta(x-y) X_s(dx) X_t(dy) ds dt,$$

where  $B \subset [0, \infty) \times [0, \infty)$  is a bounded Borel set and  $\delta$  is the Dirac delta function. Let  $D = \{(t, t) : t > 0\}$  be the time diagonal on  $\mathbb{R}_+ \times \mathbb{R}_+$ . For  $\beta = 1$ ,  $B \cap D = \emptyset$  and  $d \leq 7$ , Dynkin [6] proved the existence of SILT,  $\gamma_X$ , for a very general class of continuous superprocesses. Also, from the Dynkin's works follows the existence of SILT when  $\beta = 1$ ,  $B \cap D \neq \emptyset$  and  $d \leq 3$  (see [1]). For  $\beta = 1$ ,  $d = 4, 5$  and  $B \cap D \neq \emptyset$ , Rosen [16] proved the existence of a renormalized SILT for the  $(\alpha, d, 1)$ -superprocess. A Tanaka-like formula for the local time of  $(\alpha, d, 2)$ -superprocess was established by Adler and Lewin in [3]. The same authors derived a Tanaka-like formula for self-intersection local time for  $(\alpha, d, 2)$ -super-process (see [2]). In this paper we are going to extend the above results for the case of  $0 < \beta < 1$ .

The usual way to give a rigorous definition of SILT is to take a sequence  $(\varphi_\varepsilon)_{\varepsilon > 0}$  of smooth functions that converges in distribution to  $\delta$ , define the approximating SILTs

$$\gamma_{X,\varepsilon}(B) = \int_B \int_{\mathbb{R}^{2d}} \varphi_\varepsilon(x-y) X_s(dx) X_t(dy) ds dt, \quad B \subset \mathbb{R}_+ \times \mathbb{R}_+,$$

and prove that  $(\gamma_{X,\varepsilon}(B))_{\varepsilon > 0}$  converges, in some sense (it is usually taken  $L^2(P')$ ,  $L^{1+\beta}(P')$ ,  $L^1(P')$ , distribution or in probability), to a random variable  $\gamma_X(B)$ . In what follows we choose  $\varphi_\varepsilon = p_\varepsilon$ , where  $p_\varepsilon$  is the  $\alpha$ -stable density, given by

$$p_\varepsilon(x, y) = \frac{1}{(2\pi)^\alpha} \int_{\mathbb{R}^d} e^{-i(z \cdot (x-y)) - \varepsilon|z|^\alpha} dz, \quad x, y \in \mathbb{R}^d,$$

when  $0 < \alpha < 2$  and

$$p_\varepsilon(x, y) = \frac{1}{(2\pi\varepsilon)^{d/2}} e^{-|x-y|^2/2\varepsilon}, \quad x, y \in \mathbb{R}^d,$$

for  $\alpha = 2$ . In this paper we will consider the particular case when  $B = \{(t, s) : 0 \leq s \leq t \leq T\}$ . Here we denote  $\gamma_{X,\varepsilon}(B)$  by  $\gamma_{X,\varepsilon}(T)$ , that is

$$\gamma_{X,\varepsilon}(T) = \int_0^T \int_0^t \int_{\mathbb{R}^{2d}} p_\varepsilon(x-y) X_s(dx) X_t(dy) ds dt, \quad \forall T \geq 0.$$

Moreover, we are going to consider the renormalized SILT

$$\tilde{\gamma}_{X,\varepsilon}(T) = \gamma_{X,\varepsilon}(T) - e^{\lambda\varepsilon} \int_0^T \int_{\mathbb{R}^d} X_s(G^{\lambda,\varepsilon}(x - \cdot)) X_s(dx) ds,$$

where

$$G^{\lambda,\varepsilon}(x) = \int_\varepsilon^\infty e^{-\lambda t} p_t(x) dt, \quad \lambda, \varepsilon \geq 0.$$

Notice that  $G^{\lambda, \varepsilon}(x) \uparrow G^{\lambda, 0}(x) = G^\lambda(x)$  as  $\varepsilon \downarrow 0$  for all  $x \in \mathbb{R}^d$ , and if  $d > \alpha$  then

$$G(x) = G^{0,0}(x) = c(\alpha, d) |x|^{\alpha-d}, \quad (1.3)$$

where

$$c(\alpha, d) = \frac{\Gamma((d-\alpha)/2)}{2^{\alpha/2} \pi^{d/2} \Gamma(\alpha/2)}$$

and  $\Gamma$  is the usual Gamma function.  $G$  is called Green function of  $\Delta_\alpha$ . Also notice that, for  $\lambda > 0$ , we have

$$\Delta_\alpha G^{\lambda, \varepsilon}(x) = \lambda G^{\lambda, \varepsilon}(x) - e^{-\lambda \varepsilon} p_\varepsilon(x), \quad x \in \mathbb{R}^d. \quad (1.4)$$

Now we are ready to present our main result.

**Theorem 1.** *Let  $X$  be the  $(\alpha, d, \beta)$ -superprocess with initial measure  $X_0(dx) = \mu(dx) = h(x)dx$ , where  $h$  is bounded and integrable with respect to Lebesgue measure on  $\mathbb{R}^d$ . Let  $M$  be the martingale measure which appears in the martingale problem (1.1) for  $X$ .*

- (a) *Let  $d/2 < \alpha$ . Then there exists a process  $\gamma_X = \{\gamma_X(T) : T \geq 0\}$  such that for every  $T > 0, \delta > 0$*

$$P \left( \sup_{t \leq T} |\gamma_{X, \varepsilon}(t) - \gamma_X(t)| > \delta \right) \rightarrow 0, \text{ as } \varepsilon \downarrow 0.$$

Moreover, for any  $\lambda > 0$ ,

$$\begin{aligned} \gamma_X(T) &= \lambda \int_0^T \int_0^t \int_{\mathbb{R}^{2d}} G^\lambda(x-y) X_s(dx) X_t(dy) dt ds \\ &\quad - \int_0^T \int_{\mathbb{R}^d} X_T(G^\lambda(x-\cdot)) X_s(dx) ds \\ &\quad + \int_0^T \int_{\mathbb{R}^d} X_s(G^\lambda(x-\cdot)) X_s(dx) ds \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} G^\lambda(x-y) M(ds, dy) X_t(dx) dt, \quad \text{a.s.} \end{aligned} \quad (1.5)$$

- (b) *Let  $d/(2+(1+\beta)^{-1}) < \alpha \leq d/2$ . Then there exists a process  $\tilde{\gamma}_X = \{\tilde{\gamma}_X(T) : T \geq 0\}$  such that for every  $T > 0, \delta > 0$*

$$P \left( \sup_{t \leq T} |\tilde{\gamma}_{X, \varepsilon}(t) - \tilde{\gamma}_X(t)| > \delta \right) \rightarrow 0, \text{ as } \varepsilon \downarrow 0.$$

Moreover, for any  $\lambda > 0$ ,

$$\begin{aligned} \tilde{\gamma}_X(T) &= \lambda \int_0^T \int_0^t \int_{\mathbb{R}^{2d}} G^\lambda(x-y) X_s(dx) X_t(dy) dt ds \\ &\quad - \int_0^T \int_{\mathbb{R}^d} X_T(G^\lambda(x-\cdot)) X_s(dx) ds \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} G^\lambda(x-y) M(ds, dy) X_t(dx) dt, \quad \text{a.s.} \end{aligned} \quad (1.6)$$

The processes  $\gamma_X$  and  $\tilde{\gamma}_X$  are called SILT and renormalized SILT of  $X$ , respectively, and (1.5) and (1.6) are called Tanaka-like formula for SILT.

**Remark 1.** *It is interesting to note that our bound on dimensions*

$$d < (2 + (1 + \beta)^{-1})\alpha$$

*for renormalized SILT does not converge, as  $\beta \uparrow 1$ , to the bound  $d < 3\alpha$  established by Rosen [16] for finite variance superprocess ( $\beta = 1$ ). In fact, our bound is more restrictive, and we believe that it is related to the fact that  $(\alpha, d, \beta)$ -superprocess (with  $\beta < 1$ ) has jumps. Our conjecture is that for  $\beta < 1$ , the renormalized SILT defined by (1.6) does not exist in dimensions greater than  $(2 + (1 + \beta)^{-1})\alpha$ .*

The common ways to prove the existence of SILT for the finite variance superprocesses (see e.g. [2], [16]) do not work here. The reason for this is that such proofs strongly rely on the existence of high moments of  $X$  (at least of order four), and  $(\alpha, d, \beta)$  superprocess  $X$  has moments of order less than  $1 + \beta$ . To overcome this difficulty let us consider the path properties of  $X$  more carefully. It is well known (see Theorem 6.1.3 of [4]) that, for  $0 < \beta < 1$ , the  $(\alpha, d, \beta)$ -superprocess  $X$  is a.s. discontinuous and has jumps of the form  $\Delta X_t = r\delta_x$ , for some  $r > 0$ ,  $x \in \mathbb{R}^d$ . Here  $\delta_x$  denotes the Dirac measure concentrated at  $x$ . Let

$$N_X(dx, dr, ds) = \sum_{\{(x,r,s): \Delta X_s = r\delta_x\}} \delta_{(x,r,s)}, \quad (1.7)$$

be a random point measure on  $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+$  with compensator measure  $\hat{N}_X$  given by

$$\hat{N}_X(dx, dr, ds) = \eta r^{-2-\beta} dr X_s(dx) ds, \quad (1.8)$$

where

$$\eta = \frac{\beta(\beta + 1)}{\Gamma(1 - \beta)}.$$

Let  $K > 0$  fix. From [8] and [4] we have that the  $(\alpha, d, \beta)$ -superprocess  $X$  has the following decomposition: Let  $\varphi \in D(\Delta_\alpha)$ ,  $t \geq 0$ ,

$$\begin{aligned} X_t(\varphi) &= \mu(\varphi) + \int_0^t X_s(\Delta_\alpha \varphi) ds - C_\beta(K) \int_0^t X_s(\varphi) ds \\ &\quad + \int_0^t \int_0^K \int_{\mathbb{R}^d} r \varphi(x) \tilde{N}_X(dx, dr, ds) \\ &\quad + \int_0^t \int_K^\infty \int_{\mathbb{R}^d} r \varphi(x) N_X(dx, dr, ds), \end{aligned} \quad (1.9)$$

where  $\tilde{N}_X = N_X - \hat{N}_X$  is a martingale measure and

$$C_\beta(K) = \frac{\eta}{\beta K^\beta}.$$

As we have mentioned already, one of the problems of working with the  $(\alpha, d, \beta)$ -superprocess  $X$  is dealing with “big” jumps. In fact, the “big” jumps produce the infinite variance of the process and they appear in the term corresponding to the integral with respect to  $N_X$  on (1.9). So, the first step in the establishing the existence of SILT for  $(\alpha, d, \beta)$ -superprocess  $X$  is to “eliminate” those jumps. This is achieved via introducing the following auxiliary process.

Let us consider the canonical space,  $\Omega^\circ = D([0, \infty), M_F(\mathbb{R}^d))$ ,  $\mathcal{F}^\circ = \mathcal{B}(\Omega^\circ)$  and  $\mathcal{F}_t^\circ = \sigma\{Y_r^K : 0 \leq r \leq t\}$ , where  $Y_r^K(\omega^\circ) = \omega^\circ(r)$ . For any  $\mu \in M_F(\mathbb{R}^d)$  there exists (see [4]) a measure  $Q_\mu$  on  $(\Omega^\circ, \mathcal{F}^\circ)$ , such that

$$E_\mu [\exp(-Y_t^K(\varphi)) | \mathcal{F}_s^\circ] = \exp(-Y_s^K(V_{t-s}^K(\varphi))), \quad \forall 0 < s \leq t, \quad (1.10)$$

and

$$E_\mu [\exp(-Y_t^K(\varphi))] = \exp(-\mu(V_t^K(\varphi))), \quad \forall t > 0, \quad (1.11)$$

(notice that the expectation is taken here with respect to the measure  $Q_\mu$ ).  $V_t^K$  is the unique non-negative solution for the non-linear equation

$$\begin{aligned} \frac{\partial v_t^K}{\partial t} &= (\Delta_\alpha - C_\beta(K))v_t^K - \Phi^K(v_t^K), \\ v_0^K &= \varphi, \end{aligned} \quad (1.12)$$

where

$$\Phi^K(x) = \eta \int_0^K (e^{-ux} - 1 + ux) u^{-\beta-2} du. \quad (1.13)$$

Note that when  $K = \infty$  the resulting process  $Y^\infty$  and the regular  $(\alpha, d, \beta)$ -superprocess  $X$  have the same distribution. Now, for any  $K > 0$ , define the stopping time

$$\tau_K = \inf\{t > 0 : |\Delta X_t| > K\}. \quad (1.14)$$

In Section 2 we will show that if we define the process which evolves as  $X$  up to time  $\tau_K$  and then continues to evolve as  $Y^K$  starting at  $X_{\tau_K-}$ , then this process has the same law as  $Y^K$ . This together with the fact that  $\tau_K \uparrow \infty$  as  $K \rightarrow \infty$  (see Lemma 2) implies that it is enough to show existence of the SILT for the process  $Y^K$ . This task will be accomplished in Section 3, modulo some technical moment estimates that will be derived in Section 4. The main steps leading to the proof of Theorem 1 will be described in the next section.

## 2. PROOF OF THEOREM 1

The process  $Y^K$  whose Laplace transform is given by (1.10), (1.11) has the following decomposition:

$$\begin{aligned} Y_t^K(\varphi) &= \mu(\varphi) + \int_0^t Y_s^K(\Delta_\alpha \varphi) ds - C_\beta(K) \int_0^t Y_s^K(\varphi) ds \\ &\quad + \int_0^t \int_0^K \int_{\mathbb{R}^d} r \varphi(x) \tilde{N}_{Y^K}(dx, dr, ds), \quad \forall t \geq 0, \end{aligned} \quad (2.1)$$

where  $\tilde{N}_{Y^K} = N_{Y^K} - \hat{N}_{Y^K}$ , and

$$\begin{aligned} N_{Y^K}(dx, dr, ds) &= \sum_{\{(x,r,s): \Delta Y_s^K = r\delta_x\}} \delta_{(x,r,s)}, \\ \hat{N}_{Y^K}(dx, dr, ds) &= \eta 1_{(0,K]}(r) r^{-2-\beta} dr Y_s^K(dx) ds. \end{aligned}$$

Note that  $N_{Y^K}$  is defined in a way analogous to (1.7), however it does not have jumps “greater” than  $K$ .

In the following lemma we are going to construct the probability space where  $Y^K$  coincides with  $X$  up to the stopping time  $\tau_K$ .

**Lemma 1.** *There exists a probability space on which a pair of processes  $(\ddot{Y}^K, X)$  is defined and possesses the following properties:*

- (a)  $\ddot{Y}^K$  coincides in law with  $Y^K$ ,
- (b)  $\ddot{Y}_t^K = X_t, \quad \forall t < \tau_K$ .

*Proof.* Define

$$\Omega \equiv \Omega' \times \Omega^\circ, \quad \mathcal{F} \equiv \mathcal{F}' \times \mathcal{F}^\circ, \quad \mathcal{F}_t \equiv \mathcal{F}'_t \times \mathcal{F}^\circ_t,$$

and let

$$\ddot{Y}_t^K(w', w^\circ) \equiv \begin{cases} X_t(w'), & t < \tau_K(w'), \\ w^\circ(t - \tau_K(w')), & t \geq \tau_K(w'). \end{cases}$$

Define the measure  $P$  on  $(\Omega, \mathcal{F})$ :

$$P(B \times C) = \int_{\Omega'} 1_B(w') P^{\tau_K(w')}(C) P'(dw'),$$

where

$$P^{\tau_K(w')}(C) = Q_{X_{\tau_K(w')-}}(\{w^\circ \in \Omega^\circ : \ddot{Y}^K(w', w^\circ) \in C\}). \quad (2.2)$$

Let  $\varphi \in \text{Dom}(\Delta_\alpha)$  and  $t > 0$ . From the definition of  $\ddot{Y}^K$  we have

$$\begin{aligned} \ddot{Y}_t^K(\varphi) &= \mu(\varphi) + \int_0^t \ddot{Y}_s^K(\Delta_\alpha \varphi) ds - K_\beta \int_0^t \ddot{Y}_s^K(\varphi) ds \\ &\quad + \int_0^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_{\ddot{Y}_s^K}(dx, dr, ds), \end{aligned} \quad (2.3)$$

where  $N_{\ddot{Y}_s^K}$  is defined by (1.7) for  $t < \tau_K$  and

$$N_{\ddot{Y}_s^K}(dx, dr, ds) = \sum_{\{(x,r,s): \Delta \ddot{Y}_s^K = r\delta_x\}} \delta_{(x,r,s)},$$

for  $t \geq \tau_K$ . Let us check that  $\int_0^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_{\ddot{Y}_s^K}(dx, dr, ds)$  is an  $\mathcal{F}_t$ -martingale: For any  $t > u$ ,  $B \in \mathcal{F}'_u$ ,  $C \in \mathcal{F}^\circ_u$ , we obtain by using the definition (2.2) of  $P^{\tau_K(w')}$

$$\begin{aligned} &P \left( 1_{B \times C} \left( \int_0^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_{\ddot{Y}_s^K}(dx, dr, ds) \right. \right. \\ &\quad \left. \left. - \int_0^u \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_{\ddot{Y}_s^K}(dx, dr, ds) \right) \right) \\ &= \int_B P^{\tau_K(w')} \left( 1_C \left( \int_0^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_{\ddot{Y}_s^K}(dx, dr, ds) \right. \right. \\ &\quad \left. \left. - \int_0^u \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_{\ddot{Y}_s^K}(dx, dr, ds) \right) \right) P'(dw') \\ &= \int_B P^{\tau_K(w')} \left( 1_C \left( \int_0^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_{\ddot{Y}_s^K}(dx, dr, ds) \right. \right. \\ &\quad \left. \left. - \int_0^{(t \wedge \tau_K(w')) \vee u} \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_{\ddot{Y}_s^K}(dx, dr, ds) \right) \right) P'(dw') \\ &+ \int_B P^{\tau_K(w')} \left( 1_C \left( \int_0^{(t \wedge \tau_K(w')) \vee u} \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_{\ddot{Y}_s^K}(dx, dr, ds) \right. \right. \end{aligned}$$

$$\begin{aligned}
& - \int_0^u \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_{\tilde{Y}^K}(dx, dr, ds) \Big) \Big) P'(dw') \\
= & \int_B P^{\tau_K(w')} \left( Q_{X_{\tau_K(w')-}} [1_C \right. \\
& \quad \times \left. \int_{(t \wedge \tau_K(w')) \vee u}^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_{\tilde{Y}^K}(dx, dr, ds) | \mathcal{F}_{t \wedge \tau_K(w') \vee u}^\circ \right] \Big) P'(dw') \\
& + \int_B P^{\tau_K(w')} (1_C \\
& \quad \times \left. \int_u^{(t \wedge \tau_K(w')) \vee u} \int_0^{K-} \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_X(dx, dr, ds) \right) P'(dw') \\
= & \int_B P^{\tau_K(w')} \left( 1_C Q_{X_{\tau_K(w')-}} \left( \int_{(t \wedge \tau_K(w')) \vee u}^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_{\tilde{Y}^K}(dx, dr, ds) \right. \right. \\
& \quad \left. \left. | \mathcal{F}_{(t \wedge \tau_K(w')) \vee u}^\circ \right) \right) P'(dw') \\
& + \int_B P^{\tau_K(w')} (C \\
& \quad \times \left. \int_u^{(t \wedge \tau_K(w')) \vee u} \int_0^{K-} \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_X(dx, dr, ds) \right) P'(dw') \\
= & \int_B 1_C(X_{\cdot \wedge u}(\omega')) 1_{\{u < \tau_K\}} \\
& \quad \times P' \left( \int_u^{(t \wedge \tau_K(w')) \vee u} \int_0^{K-} \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_X(dx, dr, ds) | \mathcal{F}_u' \right) P'(dw'),
\end{aligned}$$

where in the last equality for the first term we have used the fact that for  $P'$ -a.s  $\omega'$ ,  $\tilde{N}_{\tilde{Y}^K}$  is a  $(Q_{X_{\tau_K(w')-}}, \mathcal{F}_t)$ -martingale measure on  $\mathbb{R}^d \times \mathbb{R}_+ \times [\tau_K \wedge t, t]$ . As for the second term we have used the simple identity

$$P^{\tau_K(w')}(C) 1_{\{u < \tau_K\}} = 1_C(X_{\cdot \wedge u}(\omega')) 1_{\{u < \tau_K\}}$$

for any  $C \in \mathcal{F}_u^\circ$ . Now use the fact that  $\tilde{N}_X$  is a  $(P', \mathcal{F}_t')$ -martingale measure to get that

$$P' \left( \int_u^{(t \wedge \tau_K(w')) \vee u} \int_0^{K-} \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_X(dx, dr, ds) | \mathcal{F}_u^\circ \right) = 0,$$

and the proof that  $\int_0^t \int_0^K \int_{\mathbb{R}^d} r\varphi(x) \tilde{N}_{\tilde{Y}^K}(dx, dr, ds)$  is a martingale is complete.

Then, due to the uniqueness of the decomposition ([8], Theorem 7) we conclude that  $\tilde{Y}^K$  has the same distribution as  $Y^K$ .  $\square$

**Convention.** Based on the above lemma, from now on we will assume that  $Y^K, X$  are defined on the same probability space and  $Y_t^K = X_t, \forall t < \tau_K$ .

Now we are going to show that time  $\tau_K$  can be made greater than any constant  $T$  with probability arbitrary close to 1 by taking  $K$  sufficiently large.

**Lemma 2.** *For every  $T > 0$  and  $\varepsilon > 0$ , there exists  $K > 0$  such that  $P(\tau_K \leq T) \leq \varepsilon$ .*

*Proof.* Let  $Z_T^K$  the number of jumps of height greater than  $K$  in  $[0, T] \times \mathbb{R}^d$ , that is  $Z_T^K = N([K, +\infty) \times [0, T] \times \mathbb{R}^d)$ . Then there exists (see [14], page 1430) a standard Poisson process  $A_t^K$  such that

$$Z_T^K = A_{c_\beta K^{-1-\beta} \int_0^T X_s(\mathbb{R}^d) ds}^K,$$

for some positive constant  $c_\beta$ . Then from the Markov inequality we have,

$$\begin{aligned} P(\tau_K \leq T) &= P(Z_T^K \geq 1) \\ &= P\left(A_{c_\beta K^{-1-\beta} \int_0^T X_s(\mathbb{R}^d) ds}^K \geq 1\right) \\ &= P\left(A_{c_\beta K^{-1-\beta} \int_0^T X_s(\mathbb{R}^d) ds}^K \geq 1, \frac{c_\beta}{K^{1+\beta}} \int_0^T X_s(\mathbb{R}^d) ds \geq K^{-1}\right) \\ &\quad + P\left(A_{c_\beta K^{-1-\beta} \int_0^T X_s(\mathbb{R}^d) ds}^K \geq 1, \frac{c_\beta}{K^{1+\beta}} \int_0^T X_s(\mathbb{R}^d) ds < K^{-1}\right) \\ &\leq P\left(\int_0^T X_s(\mathbb{R}^d) ds \geq c_\beta^{-1} K^\beta\right) + P(A_{K^{-1}}^K \geq 1) \\ &\leq c_\beta K^{-\beta} E_\mu \left[ \int_0^T X_s(\mathbb{R}^d) ds \right] + (1 - P(A_{K^{-1}}^K = 0)) \\ &= c_\beta T \mu(\mathbb{R}^d) K^{-\beta} + (1 - \exp(-K^{-1})). \end{aligned}$$

The result follows, since the right hand side goes to 0 as  $K \rightarrow \infty$ .  $\square$

Now the proof of Theorem 1 relies on the following proposition.

**Proposition 1.** *Let  $K > 0$  and  $Y^K$  be the truncated  $(\alpha, d, \beta)$ -superprocess with initial measure  $Y_0^K(dx) = \mu(dx) = h(x)dx$ , where  $h$  is bounded and integrable with respect to Lebesgue measure  $dx$  on  $\mathbb{R}^d$ .*

(a) *For  $d/2 < \alpha$  there exists a process  $\gamma_{Y^K}^K = \{\gamma_{Y^K}^K(T) : T \geq 0\}$  such that*

$$\lim_{\varepsilon \downarrow 0} E \left[ \sup_{t < T} \left| \gamma_{Y^K, \varepsilon}^K(t) - \gamma_{Y^K}^K(t) \right| \right] = 0, \quad \forall T > 0,$$

and for any  $\lambda > 0$ ,

$$\begin{aligned} \gamma_{Y^K}^K(T) &= \lambda \int_0^T \int_0^t \int_{\mathbb{R}^{2d}} G^\lambda(x-y) Y_s^K(dx) Y_t^K(dy) dt ds \\ &\quad - \int_0^T \int_{\mathbb{R}^d} Y_s^K(G^\lambda(x-\cdot)) Y_s^K(dx) \\ &\quad + \int_0^T \int_{\mathbb{R}^d} Y_s^K(G^\lambda(x-\cdot)) Y_s^K(dx) ds \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} G^\lambda(x-y) M^K(ds, dy) Y_t^K(dx) dt, \quad \text{a.s.} \end{aligned}$$

(b) *For  $d/(2 + (1 + \beta)^{-1}) < \alpha \leq d/2$  there exists a process  $\tilde{\gamma}_{Y^K}^K = \{\tilde{\gamma}_{Y^K}^K(T) : T \geq 0\}$  such that*

$$\lim_{\varepsilon \downarrow 0} E \left[ \sup_{t < T} \left| \tilde{\gamma}_{Y^K, \varepsilon}^K(t) - \tilde{\gamma}_{Y^K}^K(t) \right| \right] = 0, \quad \forall T > 0,$$

and for any  $\lambda > 0$ ,

$$\begin{aligned} \tilde{\gamma}_{Y^K}^K(T) &= \lambda \int_0^T \int_0^t \int_{\mathbb{R}^{2d}} G^\lambda(x-y) Y_s^K(dx) Y_t^K(dy) dt ds \\ &\quad - \int_0^T \int_{\mathbb{R}^d} Y_s^K(G^\lambda(x-\cdot)) Y_s^K(dx) ds \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} G^\lambda(x-y) M^K(ds, dy) Y_t^K(dx) dt, \quad \text{a.s.} \end{aligned}$$

Proof. Postponed.

The above proposition immediately yields:

*Proof of Theorem 1.* Fix arbitrary  $\varepsilon, \delta > 0$  and let  $d/2 < \alpha$ . Since  $X_t = Y_t^K$  for any  $t < \tau_K$ , we immediately get that

$$\gamma_X(t) = \gamma_{Y^K}(t), \quad \forall t < \tau_K,$$

and  $\gamma_X(t)$  satisfies Tanaka formula (1.5) for  $t < \tau_K$ . Moreover, since by Lemma 2,  $\tau_K \uparrow \infty$ , as  $K \rightarrow \infty$ , there is no problem to define  $\gamma_X(t)$  satisfying (1.5) for any  $t > 0$ .

Now let us check the convergence part of the theorem. For any  $T > 0$ , by Lemma 2, we can fix  $K > 0$  such that  $P(\tau_K \leq T) \leq \delta$ . Then

$$\begin{aligned} &\lim_{\varepsilon_1 \downarrow 0} P \left( \sup_{t \leq T} |\gamma_{X, \varepsilon_1}(t) - \gamma_X(t)| > \varepsilon \right) \\ &\leq \lim_{\varepsilon_1 \downarrow 0} P \left( \sup_{t \leq T} |\gamma_{X, \varepsilon_1}(t) - \gamma_X(t)| > \varepsilon, \tau_K > T \right) + P(\tau_K \leq T) \\ &\leq \lim_{\varepsilon_1 \downarrow 0} P \left( \sup_{t \leq T} |\gamma_{Y^K, \varepsilon_1}(t) - \gamma_{Y^K}(t)| > \varepsilon, \tau_K > T \right) + \delta \\ &= \delta, \end{aligned}$$

and since  $\delta, \varepsilon > 0$  were arbitrary the proof of convergence is complete.

The proof of part (b) of the theorem goes along the same lines.  $\square$

### 3. EXISTENCE OF SILT FOR $Y^K$ — PROOF OF PROPOSITION 1

Fix arbitrary  $K > 0$ . First, we derive very useful moment estimates for  $Y^K$ . Let  $\{S_t^K : t \geq 0\}$  denote the solution of the partial differential equation

$$\frac{\partial v_t^K}{\partial t} = (\Delta_\alpha - C_\beta(K))v_t^K.$$

That is,  $\{S_t^K : t \geq 0\}$  is the semigroup defined as

$$S_t^K = e^{-C_\beta(K)t} S_t. \quad (3.1)$$

Notice that  $S_t^K \varphi \leq S_t \varphi$ , for all non-negative bounded measurable functions  $\varphi$ .

Following Theorem 3.1 of [9] we have that for  $\varphi, \psi \in \mathcal{B}_b(\mathbb{R}^d)$ ,

$$E \left[ \exp \left( -Y_t^K(\varphi) - \int_0^t Y_s^K(\psi) ds \right) \middle| Y_0^K = \mu \right] = \exp(-\mu(V_t^K(\varphi, \psi))), \quad (3.2)$$

where  $\mu \in \mathcal{M}_F(\mathbb{R}^d)$  and  $V_t^K(\varphi, \psi)$  denotes the unique non-negative solution to the following evolution equation

$$v_t^K = S_t^K \varphi + \int_0^t S_s^K(\psi) ds - \int_0^t S_{t-s}^K(\Phi^K(v_s^K)) ds, \quad t \geq 0, \quad (3.3)$$

where

$$\Phi^K(x) = \sum_{m=2}^{\infty} \frac{(-1)^m}{m!} \chi(m) x^m \quad (3.4)$$

and

$$\chi(m) = \frac{\eta K^{m-1-\beta}}{m-1-\beta}. \quad (3.5)$$

Now we are going to calculate the first two moments of  $Y^K$ .

**Lemma 3.** *Let  $\varphi$  be a non-negative function on  $\mathcal{B}_b(\mathbb{R}^d)$  and  $t > 0$ . Then*

$$\begin{aligned} E_\mu [Y_t^K(\varphi)] &= \mu(S_t^K \varphi), \\ E_\mu [(Y_t^K(\varphi))^2] &= (\mu(S_t^K \varphi))^2 + \chi(2) \mu \left( \int_0^t S_{t-s}^K ((S_s^K \varphi)^2) ds \right). \end{aligned} \quad (3.6)$$

*Proof.* From (3.2) we have

$$E_\mu [e^{-\lambda Y_t^K(\varphi)}] = e^{-\mu(v_t^K(\lambda))} \quad (3.7)$$

where

$$v_t^K(\lambda) = \lambda S_t^K \varphi - \int_0^t S_{t-s}^K(\Phi^K(v_s^K(\lambda))) ds. \quad (3.8)$$

Using the elementary inequality  $e^{-x} - 1 + x \leq x^2/2$ ,  $x \geq 0$ , and (1.13) we have

$$\Phi^K(x) \leq \frac{\chi(2)}{2} x^2, \quad x \geq 0.$$

Let  $\|\cdot\|_\infty$  the supremum norm, then  $0 \leq v_t^K(\lambda) \leq \lambda S_t^K \varphi \leq \lambda \|\varphi\|_\infty$  and the previous inequality implies

$$\int_0^t S_{t-s}^K(\Phi^K(v_s^K(\lambda))) ds \leq \frac{\chi(2) \|\varphi\|_\infty^2 t}{2} \times \lambda^2.$$

Further from (3.8) we get

$$\begin{aligned} \lim_{\lambda \downarrow 0} \left| \frac{v_t^K(\lambda) - \lambda S_t^K \varphi}{\lambda} \right| &= \lim_{\lambda \downarrow 0} \frac{1}{\lambda} \int_0^t S_{t-s}^K(\Phi^K(v_s^K(\lambda))) ds \\ &\leq \lim_{\lambda \downarrow 0} \frac{\chi(2) \|\varphi\|_\infty^2 t}{2} \times \lambda = 0, \end{aligned}$$

and we write this like

$$v_t^K(\lambda) = \lambda S_t^K \varphi - o(\lambda). \quad (3.9)$$

This implies

$$\begin{aligned}
E_\mu [Y_t^K(\varphi)] &= \lim_{\lambda \downarrow 0} \frac{1 - E_\mu [e^{-\lambda Y_t^K(\varphi)}]}{\lambda} \\
&= \lim_{\lambda \downarrow 0} \frac{1 - e^{-\lambda \mu (S_t^K \varphi) + o(\lambda)}}{\lambda} \\
&= \lim_{\lambda \downarrow 0} \frac{1 - e^{-\lambda \mu (S_t^K \varphi) + o(\lambda)}}{\lambda \mu (S_t^K \varphi) - o(\lambda)} \lim_{\lambda \downarrow 0} \frac{\lambda \mu (S_t^K \varphi) - o(\lambda)}{\lambda} = \mu (S_t^K \varphi).
\end{aligned}$$

Now, to calculate the second moment we follow the ideas used in the proof of Proposition 11 of Chapter II from [11]:

$$\begin{aligned}
E_\mu [(Y_t^K(\varphi))^2] &= \lim_{\lambda \downarrow 0} \frac{2}{\lambda^2} E_\mu [e^{-\lambda Y_t^K(\varphi)} - 1 + \lambda Y_t^K(\varphi)] \\
&= \lim_{\lambda \downarrow 0} \frac{2}{\lambda^2} \left( e^{-\mu (v_t^K(\lambda))} - 1 + \lambda \mu (S_t^K \varphi) \right) \\
&= \lim_{\lambda \downarrow 0} \frac{2}{\lambda^2} \left( e^{-\lambda \mu (S_t^K \varphi) + \mu (\int_0^t S_{t-s}^K (\Phi^K(v_s^K(\lambda))) ds)} - 1 + \lambda \mu (S_t^K \varphi) \right) \\
&= \lim_{\lambda \downarrow 0} \frac{2}{\lambda^2} \left( \sum_{n=0}^{\infty} \frac{1}{n!} \left( \mu \left( \int_0^t S_{t-s}^K (\Phi^K(v_s^K(\lambda))) ds \right) - \lambda \mu (S_t^K \varphi) \right)^n \right. \\
&\quad \left. - 1 + \lambda \mu (S_t^K \varphi) \right).
\end{aligned}$$

Using the series expansion (3.4) for  $\Phi^K$  and (3.9) we obtain

$$\begin{aligned}
\int_0^t S_{t-s}^K (\Phi^K(v_s^K(\lambda))) ds &= \int_0^t S_{t-s}^K (\Phi^K(\lambda S_s^K \varphi - o(\lambda))) ds \\
&= \int_0^t S_{t-s}^K \left( \frac{\chi(2)}{2!} \lambda^2 (S_s^K \varphi)^2 + o(\lambda^2) \right) ds \\
&= \frac{\chi(2)}{2} \lambda^2 \int_0^t S_{t-s}^K ((S_s^K \varphi)^2) ds + o(\lambda^2).
\end{aligned}$$

Then

$$\begin{aligned}
E_\mu [(Y_t^K(\varphi))^2] &= \lim_{\lambda \downarrow 0} \frac{2}{\lambda^2} \left( \frac{\chi(2)}{2} \lambda^2 \mu \left( \int_0^t S_{t-s}^K ((S_s^K \varphi)^2) ds \right) + o(\lambda^2) \right. \\
&\quad \left. + \frac{1}{2!} \left( \frac{\chi(2)}{2} \lambda^2 \mu \left( \int_0^t S_{t-s}^K ((S_s^K \varphi)^2) ds \right) - \lambda \mu (S_t^K \varphi) + o(\lambda^2) \right)^2 \right) \\
&= \lim_{\lambda \downarrow 0} \frac{2}{\lambda^2} \left( \frac{\chi(2)}{2} \lambda^2 \mu \left( \int_0^t S_{t-s}^K ((S_s^K \varphi)^2) ds \right) + \frac{1}{2} \lambda^2 (\mu (S_t^K \varphi))^2 + o(\lambda^2) \right) \\
&= \chi(2) \mu \left( \int_0^t S_{t-s}^K ((S_s^K \varphi)^2) ds \right) + (\mu (S_t^K \varphi))^2,
\end{aligned}$$

and we are done.  $\square$

**Remark 2.** Using binary directed graphs, Dynkin in [6] gives a formula for the moments of superprocesses, where the Laplace functional (3.2) has an evolution equation

(3.3) with only one term  $m = 2$  in (3.4). For the  $Y^K$  superprocess it is also possible, but here the main difference is that the directed graphs are not necessarily binary.

**Corollary 1.** *Let  $\varphi, \psi$  be non-negative functions on  $\mathcal{B}_b(\mathbb{R}^d)$  and  $t \geq s > 0$ . Then*

$$\begin{aligned} E_\mu [Y_t^K(\varphi)Y_s^K(\psi)] &= \mu(S_t^K \varphi) \mu(S_s^K \psi) \\ &\quad + \chi(2)\mu \left( \int_0^s S_r^K (S_{t-r}^K \varphi S_{s-r}^K \psi) dr \right). \end{aligned}$$

*Proof.* First, use the Markov property for  $Y^K$  to get

$$E_\mu [Y_t^K(\varphi)|\mathcal{F}_s] = Y_s^K(S_{t-s}^K \varphi).$$

Therefore,

$$\begin{aligned} E_\mu [Y_t^K(\varphi)Y_s^K(\psi)] &= E_\mu [Y_s^K(S_{t-s}^K \varphi) Y_s^K(\psi)] \\ &= \frac{1}{4} \left( (E_\mu [Y_s^K(S_{t-s}^K \varphi) + Y_s^K(\psi)])^2 - (E_\mu [Y_s^K(S_{t-s}^K \varphi) - Y_s^K(\psi)])^2 \right), \end{aligned}$$

and we are done by Lemma 3.  $\square$

**Corollary 2.** *Let  $\varphi$  be non-negative functions on  $\mathcal{B}_b(\mathbb{R}^d \times \mathbb{R}^d)$  and  $t \geq s > 0$ . Then*

$$\begin{aligned} E_\mu \left[ \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x, y) Y_t^K(dx) Y_s^K(dy) \right] &= \int_{\mathbb{R}^{4d}} \mu(dx_1) \mu(dx_2) p_t(z_1 - x_1) p_s(z_2 - x_2) \varphi(z_1, z_2) dz_1 dz_2 \\ &\quad + \chi(2) \int_0^s \int_{\mathbb{R}^{4d}} \mu(dx) p_r(y - x) dy p_{t-r}(z_1 - y) p_{s-r}(z_2 - y) \varphi(z_1, z_2) dz_1 dz_2 dr. \end{aligned}$$

*Proof.* Use Corollary 1 and approximation of the  $\psi(x, y)$  by functions in the form  $\sum_i \varphi_i(x) \phi_i(y)$  to derive the result. We leave the details to the reader.  $\square$

Next proposition gives bounds on some fractional moments of  $Y^K$  and requires much more work than we have done in Lemma 3. Hence its proof will be postponed till Section 4.

**Proposition 2.** *Let  $1 + \beta < p < 2$  and  $0 < \varepsilon \leq 1$ . If*

$$d < \alpha \left( 2 + \frac{1}{p} \right),$$

*then there exists a constant  $c = c(K, p, d, \alpha, \beta)$  such that*

$$E_\mu \left[ \int_{\mathbb{R}^d} Y_t^K(p_\varepsilon(\cdot - x)) \left( \int_0^t Y_s^K(G^\lambda(\cdot - x)) ds \right)^p dx \right] < c(K, p, d, \alpha, \beta).$$

*Moreover*

$$E \left[ \int_{\mathbb{R}^d} \left( \int_0^t Y_s^K(G^\lambda(\cdot - x)) ds \right)^p Y_t^K(dx) \right] < \infty.$$

Proof. Postponed.

Now we can write the Tanaka-like formula for the approximating SILT of the truncated superprocess  $Y^K$ . From Fubini theorem, (1.4), (1.2) and (1.1) (the martingale problem for the truncated superprocess  $Y^K$ , [8]) we have

$$\begin{aligned}
\gamma_{Y^K, \varepsilon}^K(T) &= \int_0^T \int_0^t \int_{\mathbb{R}^{2d}} p_\varepsilon(x-y) Y_s^K(dx) Y_t^K(dy) ds dt \\
&= \lambda e^{\lambda \varepsilon} \int_0^T \int_0^t \int_{\mathbb{R}^{2d}} G^{\lambda, \varepsilon}(x-y) Y_s^K(dx) Y_t^K(dy) dt ds \\
&\quad - e^{\lambda \varepsilon} \int_0^T \int_{\mathbb{R}^d} \int_s^T \int_{\mathbb{R}^d} \Delta_\alpha G^{\lambda, \varepsilon}(x-y) Y_t^K(dy) dt Y_s^K(dx) ds \\
&= \lambda e^{\lambda \varepsilon} \int_0^T \int_0^t \int_{\mathbb{R}^{2d}} G^{\lambda, \varepsilon}(x-y) Y_s^K(dx) Y_t^K(dy) ds dt \\
&\quad - e^{\lambda \varepsilon} \int_0^T \int_{\mathbb{R}^d} Y_T^K(G^{\lambda, \varepsilon}(x-\cdot)) Y_s^K(dx) ds \\
&\quad + e^{\lambda \varepsilon} \int_0^T \int_{\mathbb{R}^d} Y_s^K(G^{\lambda, \varepsilon}(x-\cdot)) Y_s^K(dx) ds \\
&\quad + e^{\lambda \varepsilon} \int_0^T \int_{\mathbb{R}^d} \int_s^T \int_{\mathbb{R}^d} G^{\lambda, \varepsilon}(x-y) M^K(dt, dy) Y_s^K(dx) ds, \quad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\gamma}_{Y^K, \varepsilon}^K(T) &= \gamma_{Y^K, \varepsilon}^K(T) - e^{\lambda \varepsilon} \int_0^T \int_{\mathbb{R}^d} Y_s^K(G^{\lambda, \varepsilon}(x-\cdot)) Y_s^K(dx) ds \\
&= \lambda e^{\lambda \varepsilon} \int_0^T \int_0^t \int_{\mathbb{R}^{2d}} G^{\lambda, \varepsilon}(x-y) Y_s^K(dx) Y_t^K(dy) ds dt \\
&\quad - e^{\lambda \varepsilon} \int_0^T \int_{\mathbb{R}^d} Y_T^K(G^{\lambda, \varepsilon}(x-\cdot)) Y_s^K(dx) ds \\
&\quad + e^{\lambda \varepsilon} \int_0^T \int_{\mathbb{R}^d} \int_s^T \int_{\mathbb{R}^d} G^{\lambda, \varepsilon}(x-y) M^K(dt, dy) Y_s^K(dx) ds. \quad (3.11)
\end{aligned}$$

Note that stochastic integrals in (3.10), (3.11) are well defined due to the moment bound given by Proposition 2.

*Proof of Proposition 1.* We are going to prove Proposition 1 via letting  $\varepsilon \rightarrow 0$  in (3.10), and checking convergence of all the terms. By Corollary 2 and simple estimates we get

$$\begin{aligned}
&E \left[ \int_0^T \int_{\mathbb{R}^{4d}} Y_s^K(G^{\lambda, \varepsilon}(x-\cdot)) Y_s^K(dx) ds \right] \\
&\leq \int_0^T \int_{\mathbb{R}^{4d}} \mu(dx_1) \mu(dx_2) p_s(z_1-x_1) p_s(z_2-x_2) G^\lambda(z_1-z_2) dz_1 dz_2 ds \\
&\quad + \chi(2) \int_0^T \int_0^s \int_{\mathbb{R}^{4d}} \mu(dx) p_r(y-x) dy p_{s-r}(z_1-y) p_{s-r}(z_2-y) \\
&\quad \times G^\lambda(z_1-z_2) dz_1 dz_2 dr ds
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^T \int_{\mathbb{R}^{4d}} \mu(dx_1)\mu(dx_2) \int_0^\infty e^{-\lambda u} p_u(z_1 - z_2) p_s(z_1 - x_1) p_s(z_2 - x_2) dz_1 dz_2 du ds \\
&+ \chi(2) \int_0^T \int_0^s \int_{\mathbb{R}^{4d}} \mu(dx) p_r(y - x) dy \\
&\times \int_0^\infty e^{-\lambda u} p_u(z_1 - z_2) p_{s-r}(z_1 - y) dz_1 p_{s-r}(z_2 - y) dz_2 du dr ds \\
&\leq \int_0^T \int_0^\infty e^{-\lambda u} \int p_{u+2s}(x_1 - x_2) \mu(dx_1) \mu(dx_2) du ds \\
&+ \chi(2) \int_0^T \int_0^s \int \mu(dx) p_r(y - x) dy \int_0^\infty e^{-\lambda u} p_{u+2s-2r}(0) du dr ds \\
&\leq \|h\|_\infty \mu(1) T \lambda^{-1} + \mu(1) \chi(2) \int_0^T \int_0^s \int_0^\infty e^{-\lambda u} (u + 2s - 2r)^{-d/\alpha} du dr ds, \quad \forall T > 0,
\end{aligned} \tag{3.12}$$

where  $\|\cdot\|_\infty$  is the supremum norm and the last integral is convergent if  $d < 2\alpha$ . Using (3.12), the bound  $G^\lambda \geq G^{\lambda,\varepsilon}$  and the monotone convergence theorem to get

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} E \left[ \sup_{t < T} \left| \int_0^t \int Y_s^K (G^\lambda(x - \cdot) - G^{\lambda,\varepsilon}(x - \cdot)) Y_s^K(dx) ds \right| \right] \\
&\leq \lim_{\varepsilon \downarrow 0} E \left[ \int_0^T \int Y_s^K (G^\lambda(x - \cdot) - G^{\lambda,\varepsilon}(x - \cdot)) Y_s^K(dx) ds \right] \\
&= 0, \quad \forall T > 0.
\end{aligned} \tag{3.13}$$

In a similar way we can prove that

$$\lim_{\varepsilon \downarrow 0} E \left[ \sup_{T < L} \left| \int_0^T \int_0^t \int_{\mathbb{R}^{2d}} (G^\lambda(x - y) - G^{\lambda,\varepsilon}(x - y)) Y_s^K(dx) Y_t^K(dy) ds dt \right| \right] = 0, \tag{3.14}$$

and

$$\lim_{\varepsilon \downarrow 0} E \left[ \sup_{T < L} \left| \int_0^T \int_{\mathbb{R}^d} Y_T^K (G^\lambda(x - \cdot) - G^{\lambda,\varepsilon}(x - \cdot)) Y_s^K(dx) ds \right| \right] = 0, \tag{3.15}$$

for all  $L > 0$  and  $d < 3\alpha$ .

Now let us deal with the stochastic integral

$$\int_0^T \int F^\varepsilon(t, x) M^K(dt, dx),$$

where

$$F^\varepsilon(t, x) = \int_0^t \int_{\mathbb{R}^d} G^{\lambda,\varepsilon}(x - y) Y_s^K(dy) ds.$$

This integral is well defined if (see [12])

$$E \left[ \left( \sum_{t \in J \cap [0, T]} F^\varepsilon(t, \Delta Y_t^K)^2 \right)^{1/2} \right] < +\infty, \tag{3.16}$$

where  $J$  denotes the set of all jump times of  $X$ . Let

$$d < \alpha \left( 2 + \frac{1}{1 + \beta} \right), \quad (3.17)$$

hence we can choose  $p \in (1 + \beta, 2)$  such that

$$d < \alpha \left( 2 + \frac{1}{p} \right). \quad (3.18)$$

Since  $p \in (1 + \beta, 2)$  we can use the Jensen inequality to get

$$\left( \sum_{i \in I} a_i \right)^{p/2} \leq \sum_{i \in I} a_i^{p/2}$$

if  $a_i \geq 0$  for all  $i \in I$ . This yields

$$\begin{aligned} & E \left[ \left( \sum_{t \in J \cap [0, T]} F^\varepsilon(t, \Delta Y_t^K)^2 \right)^{1/2} \right] \\ & \leq \left( E \left[ \sum_{t \in J \cap [0, T]} F^\varepsilon(t, \Delta Y_t^K)^p \right] \right)^{1/p} \\ & = \left( E \left[ \eta \int_0^T \int \int_0^K u^{-\beta-2} (F^\varepsilon(t, u\delta_x))^p du Y_t^K(dx) dt \right] \right)^{1/p} \\ & = c \left( \int_0^T E \left[ \int \left( \int_0^t \int G^{\lambda, \varepsilon}(x-y) Y_s^K(dy) ds \right)^p Y_t^K(dx) \right] dt \right)^{1/p}. \end{aligned}$$

Since  $p$  satisfies (3.18), the condition (3.16) follows from Proposition 2.

Let  $F = F^0$ . By Burkholder-Davis-Gundy inequality (see [12]) and the previous argument we get

$$\begin{aligned} & E \left[ \sup_{t \leq T} \left| \int_0^t \int (F(s, x) - F^\varepsilon(s, x)) M^K(ds, dx) \right| \right] \\ & \leq E \left[ \sup_{t \leq T} \int_0^t \int (|F(s, x) - F^\varepsilon(s, x)|) M^K(ds, dx) \right] \\ & \leq cE \left[ \left( \sum_{t \in J \cap [0, T]} ((F - F^\varepsilon)(t, \Delta Y_t^K))^2 \right)^{1/2} \right] \\ & \leq c \left( \int_0^T E \left[ \int \left( \int_0^t \int |(G^\lambda - G^{\lambda, \varepsilon})(x-y)| Y_s^K(dy) ds \right)^p Y_t^K(dx) \right] dt \right)^{1/p} \\ & \rightarrow 0, \text{ as } \varepsilon \downarrow 0, \forall T > 0, \end{aligned} \quad (3.19)$$

where the last convergence follows by Proposition 2 and the monotone convergence theorem. Now combine (3.13), (3.14), (3.15), (3.17) and (3.19) to get that all the terms in (3.10) converge and the proof of part **(a)** is complete. By (3.14), (3.15), (3.17) and (3.19) we get that all the terms in (3.11) converge and hence the part **(b)** of the proposition follows.  $\square$

## 4. PROOF OF PROPOSITION 2: ESTIMATION OF FRACTIONAL MOMENTS

In what follows we will use the following well known equalities. For  $p \in (1, 2)$

$$z^{p-1} = \eta_p \int_0^\infty (1 - e^{-\lambda z}) \lambda^{-p} d\lambda \quad (4.1)$$

and

$$z^p = p\eta_p \int_0^\infty (e^{-\lambda z} - 1 + \lambda z) \lambda^{-p-1} d\lambda \quad (4.2)$$

where

$$\eta_p = \frac{p-1}{\Gamma(2-p)}.$$

**Proposition 3.** *Let  $1 < 1 + \beta < p < p' < 2$ . Then there exists a constant  $c = c(K, \beta, p, p')$  such that for any non-negative functions  $\varphi, \psi \in \mathcal{B}_b(\mathbb{R}^d)$ ,*

$$\begin{aligned} & E \left[ Y_t^K(\varphi) \left( \int_0^t Y_s^K(\psi) ds \right)^p \middle| Y_0^K = \mu \right] \\ & \leq c \left\{ \mu (S_t^K \varphi) + \mu (S_t^K \varphi) \left( \mu \left( \int_0^t S_s^K \psi ds \right) \right)^p \right. \\ & \quad + \mu (S_t^K \varphi) \mu \left( \int_0^t S_{t-s}^K \left( \left( \int_0^s S_r^K \psi dr \right)^{p'} \right) ds \right) \\ & \quad + \mu \left( \int_0^t S_{t-s}^K \left( S_s^K \varphi \int_0^s S_r^K \psi dr \right) ds \right) \\ & \quad + \mu \left( \int_0^t S_{t-s}^K \left( S_s^K \varphi \int_0^s S_r^K \psi dr \right) ds \right) \left( \mu \left( \int_0^t S_s^K \psi ds \right) \right)^{p-1} \\ & \quad + \mu \left( \int_0^t S_{t-s}^K \left( \int_0^s S_{s-r}^K \left( S_r^K \varphi \int_0^r S_u^K \psi du \right) dr \right) \left( \int_0^s S_r^K \psi dr \right)^{p-1} \right) ds \\ & \quad + \mu \left( \int_0^t S_{t-s}^K \left( S_s^K \varphi \left( \int_0^s S_r^K \psi dr \right)^p \right) ds \right) \\ & \quad \left. + \mu \left( \int_0^t S_{t-s}^K \left( S_s^K \varphi \int_0^s S_{s-r}^K \left( \left( \int_0^r S_u^K \psi du \right)^{p'} \right) dr \right) ds \right) \right\}, \quad \forall t > 0. \quad (4.3) \end{aligned}$$

*Proof.* Fix an arbitrary  $t > 0$ . By (4.2) we obtain

$$\begin{aligned} & E_\mu \left[ Y_t^K(\varphi) \left( \int_0^t Y_s^K(\psi) ds \right)^p \right] \\ & = p\eta_p \int_0^\infty \lambda^{-p-1} \left( E \left[ Y_t^K(\varphi) e^{-\lambda \int_0^t Y_s^K(\psi) ds} \right] \right. \\ & \quad \left. - E \left[ Y_t^K(\varphi) \right] + \lambda E \left[ Y_t^K(\varphi) \int_0^t Y_s^K(\psi) ds \right] \right) d\lambda. \quad (4.4) \end{aligned}$$

Now we will bound the moments on the right hand side of the above expression. First of all, by Corollary 1, we have

$$\begin{aligned} E_\mu \left[ Y_t^K(\varphi) \int_0^t Y_s^K(\psi) ds \right] &= \mu(S_t^K \varphi) \mu \left( \int_0^t S_s^K \psi ds \right) \\ &+ \chi(2) \mu \left( \int_0^t \int_0^s S_r^K(S_{t-r}^K \varphi S_{s-r}^K \psi) dr ds \right). \end{aligned} \quad (4.5)$$

Moreover, from Fubini theorem we get the following useful equality

$$\int_0^t \int_0^s S_r^K(S_{t-r}^K \varphi S_{s-r}^K \psi) dr ds = \int_0^t S_{t-r}^K \left( S_r^K \varphi \int_0^r S_s^K \psi ds \right) dr. \quad (4.6)$$

Now let us estimate the remaining moment. Use the Laplace transform (3.2) and the dominated convergence theorem to obtain

$$\begin{aligned} E_\mu \left[ Y_t^K(\varphi) e^{-\lambda \int_0^t Y_s^K(\psi) ds} \right] &= - \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E_\mu \left[ e^{-\lambda \int_0^t Y_s^K(\psi) ds - \varepsilon Y_t^K(\varphi)} - e^{-\lambda \int_0^t Y_s^K(\psi) ds} \right] \\ &= - \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left( e^{-\mu(V_t^K(\varepsilon \varphi, \lambda \psi))} - e^{-\mu(V_t^K(0, \lambda \psi))} \right). \end{aligned}$$

From (3.2), we can easily derive that

$$V_s^K(\varphi, \psi) \geq V_s^K(0, \psi) \geq 0, \quad \forall s \geq 0, \quad (4.7)$$

and hence by the dominated convergence theorem, we get

$$V_t^K(\varepsilon \varphi, \lambda \psi) \rightarrow V_t^K(0, \lambda \psi), \quad \text{as } \varepsilon \downarrow 0.$$

Using the same argument we get

$$E \left[ Y_t^K(\varphi) e^{-\lambda \int_0^t Y_s^K(\psi) ds} \right] = e^{-\mu(V_t^K(0, \lambda \psi))} \mu \left( \lim_{\varepsilon \downarrow 0} \frac{V_t^K(\varepsilon \varphi, \lambda \psi) - V_t^K(0, \lambda \psi)}{\varepsilon} \right).$$

Following the argument in Section 6.3 of [5] we can show that  $U_t^K(\varphi, \lambda \psi)$  defined by

$$U_t^K(\varphi, \lambda \psi) = \lim_{\varepsilon \downarrow 0} \frac{V_t^K(\varepsilon \varphi, \lambda \psi) - V_t^K(0, \lambda \psi)}{\varepsilon}.$$

satisfies the following equation,

$$U_t^K(\varphi, \lambda \psi) = S_t^K \varphi - \int_0^t S_{t-s}^K (U_s^K(\varphi, \lambda \psi) (\Phi^K)'(V_s^K(0, \lambda \psi))) ds, \quad (4.8)$$

with  $\Phi'(x) = \frac{d\Phi(x)}{dx}$ .

Use (3.6), (4.4), (4.5), (4.6) and (4.8) to get

$$\begin{aligned}
& E \left[ Y_t^K(\varphi) \left( \int_0^t Y_s^K(\psi) ds \right)^p \right] \\
&= p\eta_p \int_0^\infty \left( e^{-\mu(V_t^K(0, \lambda\psi))} \mu(U_t^K(\varphi, \lambda\psi)) + \lambda \mu(S_t^K \varphi) \mu \left( \int_0^t S_s^K \psi ds \right) \right. \\
&\quad \left. - \mu(S_t^K \varphi) + \lambda \chi(2) \mu \left( \int_0^t S_{t-s}^K(S_s^K \varphi \int_0^s S_r^K \psi dr) ds \right) \right) \lambda^{-p-1} d\lambda \\
&= p\eta_p \mu \left( \int_0^\infty \left( S_t^K \varphi e^{-\lambda \mu(\int_0^t S_s^K \psi ds)} - S_t^K \varphi + \lambda S_t^K \varphi \mu \left( \int_0^t S_s^K \psi ds \right) \right. \right. \\
&\quad \left. \left. + S_t^K \varphi e^{-\mu(V_t^K(0, \lambda\psi))} - S_t^K \varphi e^{-\lambda \mu(\int_0^t S_s^K \psi ds)} \right. \right. \\
&\quad \left. \left. + \lambda \chi(2) \int_0^t S_{t-s}^K(S_s^K \varphi \int_0^s S_r^K \psi dr) ds \right. \right. \\
&\quad \left. \left. - e^{-\mu(V_t^K(0, \lambda\psi))} \int_0^t S_{t-s}^K(U_s^K(\varphi, \lambda\psi)(\Phi^K)'(V_s^K(0, \lambda\psi))) ds \right) \lambda^{-p-1} d\lambda \right) \\
&= \mu(S_t^K \varphi) \left( \mu \left( \int_0^t S_s^K \psi ds \right) \right)^p + I_1 + I_2 + I_3
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= p\eta_p \mu \left( S_t^K \varphi \int_0^\infty \left( e^{-\mu(V_t^K(0, \lambda\psi))} - e^{-\lambda \mu(\int_0^t S_s^K \psi ds)} \right) \lambda^{-p-1} d\lambda \right), \\
I_2 &= p\eta_p \mu \left( \int_0^\infty \left( \lambda \chi(2) \int_0^t S_{t-s}^K(S_s^K \varphi \int_0^s S_r^K \psi dr) ds \right. \right. \\
&\quad \left. \left. - \int_0^t S_{t-s}^K(U_s^K(\varphi, \lambda\psi)(\Phi^K)'(V_s^K(0, \lambda\psi))) ds \right) \lambda^{-p-1} d\lambda \right), \\
I_3 &= p\eta_p \mu \left( \int_0^\infty \left( 1 - e^{-\mu(V_t^K(0, \lambda\psi))} \right) \right. \\
&\quad \left. \times \int_0^t S_{t-s}^K(U_s^K(\varphi, \lambda\psi)(\Phi^K)'(V_s^K(0, \lambda\psi))) ds \lambda^{-p-1} d\lambda \right).
\end{aligned}$$

By the elementary inequality  $1 - e^{-x} \leq x$ , for  $x \geq 0$ , and (3.5) we have

$$(\Phi^K)'(x) \leq \eta \frac{K^{1-\beta}}{1-\beta} x = \chi(2)x. \quad (4.9)$$

Using (3.3) and (4.7) it is easy to derive that  $U_t^K(\varphi, \lambda\psi) \geq 0$  and

$$U_t^K(\varphi, \lambda\psi) \leq S_t^K \varphi, \quad (4.10)$$

$$V_t^K(0, \lambda\psi) \leq \lambda \int_0^t S_s^K \psi ds. \quad (4.11)$$

The above inequalities and (4.8) yield the following bound on  $I_3$ :

$$\begin{aligned}
I_3 &\leq c\mu \left( \int_0^\infty \int_0^t S_{t-s}^K (S_s^K \varphi V_s^K(0, \lambda\psi)) ds \left( 1 - e^{-\mu(V_t^K(0, \lambda\psi))} \right) \lambda^{-p-1} d\lambda \right) \\
&\leq c\mu \left( \int_0^t S_{t-s}^K (S_s^K \varphi \int_0^s S_r^K \psi dr) ds \right) \int_0^\infty \left( 1 - e^{-\lambda\mu \int_0^t S_r^K \psi dr} \right) \lambda^{-p} d\lambda \\
&\leq c \left( \mu \left( \int_0^t S_s^K \psi ds \right) \right)^{p-1} \mu \left( \int_0^t S_{t-s}^K (S_s^K \varphi \int_0^s S_r^K \psi dr) ds \right), \quad (4.12)
\end{aligned}$$

where the last inequality follows by (4.1).

Let us take care of  $I_2$ . By (4.8) we get

$$\begin{aligned}
I_2 &= p\eta_p\mu \left( \int_0^\infty \left( \lambda\chi(2) \int_0^t S_{t-s}^K (S_s^K \varphi \int_0^s S_r^K \psi dr) ds \right. \right. \\
&\quad \left. \left. - \int_0^t S_{t-s}^K (S_s^K \varphi (\Phi^K)'(V_s^K(0, \lambda\psi))) ds \right. \right. \\
&\quad \left. \left. + \int_0^t S_{t-s}^K ((\Phi^K)'(V_s^K(0, \lambda\psi))) \right. \right. \\
&\quad \left. \left. \times \int_0^s S_{s-r}^K (U_r^K(\varphi, \lambda\psi) (\Phi^K)'(V_r^K(0, \lambda\psi))) dr \right) ds \right) \lambda^{-p-1} d\lambda \\
&= J_1 + J_2, \quad (4.13)
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= p\eta_p\mu \left( \int_0^\infty \left( \lambda\chi(2) \int_0^t S_{t-s}^K (S_s^K \varphi \int_0^s S_r^K \psi dr) ds \right. \right. \\
&\quad \left. \left. - \int_0^t S_{t-s}^K (S_s^K \varphi (\Phi^K)'(V_s^K(0, \lambda\psi))) ds \right) \lambda^{-p-1} d\lambda \right), \\
J_2 &= p\eta_p\mu \left( \int_0^\infty \int_0^t S_{t-s}^K ((\Phi^K)'(V_s^K(0, \lambda\psi))) \right. \\
&\quad \left. \times \int_0^s S_{s-r}^K (U_r^K(\varphi, \lambda\psi) (\Phi^K)'(V_r^K(0, \lambda\psi))) dr \right) ds \lambda^{-p-1} d\lambda.
\end{aligned}$$

Let us estimate is  $J_2$ . First, by (1.13), (4.1) and (4.2) we obtain

$$\begin{aligned}
\int_0^\infty (\Phi^K)'(V_s^K(0, \lambda\psi)) \lambda^{-p} d\lambda &= \eta \int_0^\infty \int_0^K \left( 1 - e^{-wV_s^K(0, \lambda\psi)} \right) w^{-\beta-1} dw \lambda^{-p} d\lambda \\
&\leq \eta \int_0^K \int_0^\infty \left( 1 - e^{-\lambda w \int_0^s S_r^K \psi dr} \right) \lambda^{-p} d\lambda w^{-\beta-1} dw \\
&= \eta\eta_p^{-1} \int_0^K \left( w \int_0^s S_r^K \psi dr \right)^{p-1} w^{-\beta-1} dw \\
&= c \left( \int_0^s S_r^K \psi dr \right)^{p-1}.
\end{aligned}$$

Use this and (4.9), (4.10), (4.11) to get

$$\begin{aligned}
J_2 &\leq c\mu \left( \int_0^\infty \int_0^t S_{t-s}^K (\Phi^K)'(V_s^K(0, \lambda\psi)) \right. \\
&\quad \times \left. \int_0^s S_{s-r}^K (S_r^K \varphi V_r^K(0, \lambda\psi)) dr ds \lambda^{-p-1} d\lambda \right) \\
&\leq c\mu \left( \int_0^t S_{t-s}^K \left( \int_0^s S_{s-r}^K (S_r^K \varphi \int_0^r S_u^K \psi du) dr \right. \right. \\
&\quad \times \left. \left. \int_0^\infty (\Phi^K)'(V_s^K(0, \lambda\psi)) \lambda^{-p} d\lambda \right) ds \right) \\
&\leq c\mu \left( \int_0^t S_{t-s}^K \left( \int_0^s S_{s-r}^K \left( S_r^K \varphi \int_0^r S_u^K \psi du \right) dr \left( \int_0^s S_l^K \psi dl \right)^{p-1} \right) ds \right). \tag{4.14}
\end{aligned}$$

Now let us estimate  $J_1$  :

$$\begin{aligned}
J_1 &= p\eta_p \mu \left( \int_0^\infty \left( \lambda\chi(2) \int_0^t S_{t-s}^K \left( S_s^K \varphi \int_0^s S_r^K \psi dr \right) \right. \right. \\
&\quad \left. \left. - \int_0^t S_{t-s}^K \left( S_s^K \varphi \eta \int_0^K (1 - e^{-wV_s^K(0, \lambda\psi)}) w^{-\beta-1} dw \right) ds \right) \lambda^{-p-1} d\lambda \right) \\
&= p\eta_p \mu \left( \int_0^t S_{t-s}^K \left( S_s^K \varphi \left( \int_0^\infty \left[ \lambda\chi(2) \int_0^s S_r^K \psi dr \right. \right. \right. \right. \\
&\quad \left. \left. \left. + \eta \int_0^K (e^{-wV_s^K(0, \lambda\psi)} - 1) w^{-\beta-1} dw \right] \lambda^{-p-1} d\lambda \right) ds \right).
\end{aligned}$$

Using the identity

$$\chi(2) = \eta \int_0^K w^{-\beta} dw,$$

we obtain

$$\begin{aligned}
J_1 &= p\eta_p \mu \left( \int_0^t S_{t-s}^K (S_s^K \varphi \eta \int_0^\infty \int_0^K (e^{-wV_s^K(0, \lambda\psi)} - 1 \right. \\
&\quad \left. + \lambda w \int_0^s S_r^K \psi dr) w^{-\beta-1} dw \lambda^{-p-1} d\lambda \right) ds \\
&= p\eta_p \mu \left( \int_0^t S_{t-s}^K (S_s^K \varphi \left[ \int_0^K \int_0^\infty (e^{-\lambda w \int_0^s S_r^K \psi dr} - 1 \right. \right. \\
&\quad \left. \left. + \lambda w \int_0^s S_r^K \psi dr) \lambda^{-p-1} d\lambda w^{-\beta-1} dw \right. \right. \\
&\quad \left. \left. + \int_0^K \int_0^\infty (e^{-wV_s^K(0, \lambda\psi)} - e^{-\lambda w \int_0^s S_r^K \psi dr}) \lambda^{-p-1} d\lambda w^{-\beta-1} dw \right] \right) ds
\end{aligned}$$

$$\begin{aligned}
&= \mu \left( \int_0^t S_{t-s}^K \left( S_s^K \varphi \eta \int_0^K \left( w \int_0^s S_r^K \psi dr \right)^p w^{-\beta-1} dw \right) \right) \\
&\quad + p\eta\eta_p \mu \left( \int_0^t S_{t-s}^K (S_s^K \varphi \right. \\
&\quad \times \left. \int_0^K \int_0^\infty \left( e^{-wV_s^K(0, \lambda\psi)} - e^{-\lambda w \int_0^s S_r^K \psi dr} \right) \lambda^{-p-1} d\lambda w^{-\beta-1} dw \right) ds \Big) \\
&= c\mu \left( \int_0^t S_{t-s}^K \left( S_s^K \varphi \left( \int_0^s S_r^K \psi dr \right)^p \right) \right) ds \\
&\quad + p\eta\eta_p \mu \left( \int_0^t S_{t-s}^K (S_s^K \varphi Q(s)) ds \right), \tag{4.15}
\end{aligned}$$

where

$$\begin{aligned}
Q(s) &= \int_0^K \int_0^\infty \left( e^{-wV_s^K(0, \lambda\psi)} - e^{-\lambda w \int_0^s S_r^K \psi dr} \right) \lambda^{-p-1} d\lambda w^{-\beta-1} dw \\
&\leq \int_0^K \int_0^\infty \left| wV_s^K(0, \lambda\psi) - \lambda w \int_0^s S_r^K \psi dr \right| \lambda^{-p-1} d\lambda w^{-\beta-1} dw \\
&= c \left( \int_0^K + \int_K^\infty \right) \left| V_s^K(0, \lambda\psi) - \int_0^s S_r^K(\lambda\psi) dr \right| \lambda^{-p-1} d\lambda \\
&= (Q_1 + Q_2)(s).
\end{aligned}$$

By (3.3)

$$Q_1(s) = c \int_0^K \int_0^s S_{s-r}^K \Phi^K(V_r^K(0, \lambda\psi)) dr \lambda^{-p-1} d\lambda.$$

Fix an arbitrary  $p'$  such that  $1 < p < p' < 2$ . Then, from the elementary inequality  $0 \leq e^{-x} - 1 - x \leq cx^{p'}$  for  $x \geq 0$ , we obtain

$$\Phi^K(x) \leq c\eta \frac{K^{p'-\beta-1}}{p' - \beta - 1} x^{p'}.$$

Use this to get

$$\begin{aligned}
Q_1(s) &\leq c \int_0^K \int_0^s S_{s-r}^K \left( (V_r^K(0, \lambda\psi))^{p'} \right) dr \lambda^{-p-1} d\lambda \\
&\leq c \int_0^K \int_0^s S_{s-r}^K \left( \left( \int_0^r S_u^K(\lambda\psi) du \right)^{p'} \right) dr \lambda^{-p-1} d\lambda \\
&= c \int_0^s S_{s-r}^K \left( \left( \int_0^r S_u^K \psi du \right)^{p'} \right) dr. \tag{4.16}
\end{aligned}$$

Apply triangle inequality and (4.11) to bound  $Q_2$ :

$$\begin{aligned}
Q_2(s) &= c \int_K^\infty \left| V_s^K(0, \lambda\psi) - \int_0^s S_r^K(\lambda\psi) dr \right| \lambda^{-p-1} d\lambda \\
&\leq 2c \int_K^\infty \int_0^s S_r^K(\lambda\psi) dr \lambda^{-p-1} d\lambda \\
&= c \int_0^s S_r^K \psi dr. \tag{4.17}
\end{aligned}$$

Finally, let us estimate  $I_1$ . Proceeding as before we have

$$\begin{aligned}
I_1 &= p\eta_p\mu \left( S_t^K \varphi \left( \int_0^K + \int_K^\infty \right) \left( e^{-\mu(V_t^K(0, \lambda\psi))} - e^{-\lambda\mu(\int_0^t S_s^K \psi ds)} \right) \lambda^{-p-1} d\lambda \right) \\
&\leq p\eta_p\mu \left( S_t^K \varphi \int_0^K \left| \mu(V_t^K(0, \lambda\psi)) - \mu \left( \int_0^t S_s^K \psi ds \right) \right| \lambda^{-p-1} d\lambda \right) \\
&\quad + p\eta_p\mu \left( S_t^K \varphi \int_K^\infty 2\lambda^{-p-1} d\lambda \right) \\
&\leq c\mu(S_t^K \varphi) \int_0^K \mu \left( \int_0^t S_{t-s}^K (V_s^K(0, \lambda\psi))^{p'} ds \right) \lambda^{-p-1} d\lambda + c\mu(S_t^K \varphi) \\
&= c\mu(S_t^K \varphi) \mu \left( \int_0^t S_{t-s}^K \left( \int_0^s S_r^K \psi dr \right)^{p'} ds \right) + c\mu(S_t^K \varphi). \tag{4.18}
\end{aligned}$$

Combining (4.12)-(4.18) and (3.6) we obtain (4.3).  $\square$

Now, the proof of Proposition 2 is based on the bounds that we will get on all the terms on the right hand side of (4.3).

**Lemma 4.** *Let  $\mu(dx) = h(x)dx$ , where  $h$  is bounded and integrable. Then*

$$\sup_{x \in \mathbb{R}^d} \mu(S_s G^\lambda(\cdot \cdot - x)) \leq c_{4.19}(h, \lambda) < \infty. \tag{4.19}$$

*Proof.* Using the explicit expression for  $\mu$  we have,

$$\begin{aligned}
\mu(S_s G^\lambda(\cdot \cdot - x)) &= \int \int p_s(y-z) G^\lambda(z-x) dz \mu(dy) \\
&\leq \int \int p_s(y-z) G^\lambda(z-x) dz \|h\|_\infty dy \\
&= \|h\|_\infty \|G^\lambda\|_1 = \|h\|_\infty \lambda^{-1},
\end{aligned}$$

recall that  $\|\cdot\|_\infty$  is the supremum norm.  $\square$

In the next two lemmas we are going to use the following basic inequalities: For  $d > \alpha$  and  $\delta \in (0, 1)$ , we have ([13], Lemma 4)

$$p_t(x) \leq ct^{\delta-1} |x|^{\alpha-d-\alpha\delta}, \quad t > 0, x \in \mathbb{R}^d, \tag{4.20}$$

and the Riesz convolution formula

$$\int_{\mathbb{R}^d} |x-z|^{a-d} |z-y|^{b-d} dz = c|x-y|^{a+b-d}, \tag{4.21}$$

whenever  $a, b > 0, a+b < d$  and  $x, y \in \mathbb{R}^d$ .

Also define the indicator function:

$$\kappa(x) = \mathbf{1}(|x| \leq 1).$$

**Lemma 5.** *Let  $\alpha < d$ . Then, for any  $\delta \in (0, 1)$  and  $a \in [0, d)$ , we have*

$$\int_{\mathbb{R}^d} p_s(y-z) |z-x|^{-a} dz \leq c_1 + c_2 s^{\delta-1} (|y-x|^{\alpha-a-\delta\alpha} \kappa(y-x) + 1), \quad \forall y, x \in \mathbb{R}^d, \tag{4.22}$$

where  $c_1 \geq 1, c_2 > 0$  are constants.

*Proof.* First let us prove (4.22) for the case  $\alpha - \delta\alpha < a$ . Use (4.20) and (4.21) to get

$$\begin{aligned}
\int_{\mathbb{R}^d} p_s(y-z)|z-x|^{-a} dz &= \int_{\mathbb{R}^d} p_s(y-x-z)|z|^{-a} dz \\
&= \left( \int_{|z|>1} + \int_{|z|\leq 1} \right) p_s(y-x-z)|z|^{-a} dz \\
&\leq 1 + \int_{|z|\leq 1} s^{\delta-1}|y-x-z|^{\alpha-d-\delta\alpha}|z|^{-a} dz \\
&\leq 1 + cs^{\delta-1}|y-x|^{\alpha-a-\delta\alpha} \\
&= 1 + cs^{\delta-1}(|y-x|^{\alpha-a-\delta\alpha}\mathbf{1}(|y-x|\leq 1) \\
&\quad + |y-x|^{\alpha-a-\delta\alpha}\mathbf{1}(|y-x|>1)) \\
&\leq 1 + cs^{\delta-1}(|y-x|^{\alpha-a-\delta\alpha}\kappa(y-x) + 1).
\end{aligned}$$

Now, suppose  $\alpha - \delta\alpha \geq a$ . Using a simple coupling argument, as in Lemma 5.1 of [15], we have

$$\int_{\mathbb{R}^d} p_s(y-x-z)|z|^{-a} dz \leq \int_{\mathbb{R}^d} p_s(z)|z|^{-a} dz.$$

By the scaling relationship

$$p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha}x), \quad t > 0, x \in \mathbb{R}^d, \quad (4.23)$$

we get

$$\begin{aligned}
\int_{\mathbb{R}^d} p_s(z)|z|^{-a} dz &= s^{-a/\alpha} \int_{\mathbb{R}^d} p_1(s^{-1/\alpha}z)|s^{-1/\alpha}z|^{-a} s^{-d/\alpha} dz \\
&= s^{-a/\alpha} \int_{\mathbb{R}^d} p_1(z)|z|^{-a} dz.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^d} p_s(y-z)|z-x|^{-a} dz &\leq cs^{-a/\alpha}(\mathbf{1}(|s|\leq 1) + \mathbf{1}(|s|>1)) \\
&\leq cs^{\delta-1} + c,
\end{aligned}$$

and we are done.  $\square$

**Lemma 6.** For any  $\delta \in (0, 1)$  there exists  $c(t)$  such that for any  $T > 0$ ,  $\sup_{t < T} c(t) < \infty$  and

$$\int_0^t S_s G^\lambda(\cdot - x)(y) ds \leq c(t)(|y-x|^{2\alpha-d-\delta\alpha}\kappa(y-x) + 1), \quad \forall y, x \in \mathbb{R}^d. \quad (4.24)$$

*Proof.* Let  $\alpha < d$ . Since  $G^\lambda(x) \leq c|x|^{\alpha-d}$ , take  $a = d - \alpha$ , apply Lemma 5 and make additional integration with respect to time. If  $\alpha \geq d$ , then by the unimodality of  $p_1$ ,

$$\begin{aligned}
\int_0^t S_s G^\lambda(\cdot - x)(y) ds &= \int_0^t \int_0^\infty e^{-\lambda r} p_{r+s}(x-y) dr ds \\
&\leq c \int_0^t \int_0^\infty e^{-\lambda r} (r+s)^{-d/\alpha} dr ds \\
&\leq c(t), \quad \forall t \geq 0,
\end{aligned}$$

and we are done.  $\square$

**Lemma 7.** *Let  $1 < q < 2$ , and  $d < \alpha(2 + 1/q)$ . Then there exists  $c(t)$  such that for any  $T > 0$ ,  $\sup_{t < T} c(t) < \infty$  and for any  $y, x \in \mathbb{R}^d$ ,*

$$\int_0^t S_{t-s} \left( \left( \int_0^s S_r G^\lambda(\cdot - x) dr \right)^q \right) (y) ds \leq c(t), \quad (4.25)$$

$$\int_0^t S_{s+\varepsilon} \left( \left( \int_0^s S_r G^\lambda(\cdot - x) dr \right)^q \right) (y) ds \leq c(t), \quad \forall \varepsilon \in [0, 1]. \quad (4.26)$$

*Proof.* Since  $d < \alpha(2 + 1/q)$  it is easy to check that we can fix  $\delta \in (0, 1)$  sufficiently small such that,

$$q(2\alpha - d - \delta\alpha) + \alpha - \delta\alpha > 0. \quad (4.27)$$

By Lemma 6,

$$\left( \int_0^s S_r G^\lambda(\cdot - x) dr \right)^q (y) \leq \sup_{s \leq t} c(s) (|y - x|^{q(2\alpha - d - \delta\alpha)} \kappa(y - x) + 1). \quad (4.28)$$

Now take  $a = -q(2\alpha - d - \delta\alpha)$ . If  $a < 0$  then the result follows trivially, due to the fact that then the right hand side of (4.28) is uniformly bounded for any  $y, x \in \mathbb{R}^d$ .

If  $a \geq 0$ , we apply again Lemma 5 to conclude that the result follows if  $q(2\alpha - d - \delta\alpha) + \alpha - \delta\alpha > 0$ . But this is exactly the condition (4.27) which is satisfied due to the choice of  $\delta$ .  $\square$

*Proof of Proposition 2.* From (3.1) we see that  $S_t^K \leq S_t$ , hence Proposition 3 implies

$$\begin{aligned} & E_\mu \left[ \int_{\mathbb{R}^d} Y_t^K(p_\varepsilon(\cdot - x)) \left( \int_0^t Y_s^K(G^\lambda(\cdot \cdot - x)) ds \right)^p dx \right] \\ & \leq c \left\{ \int_{\mathbb{R}^d} \mu(S_t p_\varepsilon(\cdot - x)) dx \right. \\ & \quad + \int_{\mathbb{R}^d} \mu(S_t p_\varepsilon(\cdot - x)) \left( \mu \left( \int_0^t S_s G^\lambda(\cdot \cdot - x) ds \right) \right)^p dx \\ & \quad + \int_{\mathbb{R}^d} \mu(S_t p_\varepsilon(\cdot - x)) \mu \left( \int_0^t S_{t-s} \left( \int_0^s S_r G^\lambda(\cdot \cdot - x) dr \right)^{p'} ds \right) dx \\ & \quad + \int_{\mathbb{R}^d} \mu \left( \int_0^t S_{t-s} \left( S_s p_\varepsilon(\cdot - x) \int_0^s S_r G^\lambda(\cdot \cdot - x) dr \right) ds \right) dx \\ & \quad + \int_{\mathbb{R}^d} \mu \left( \int_0^t S_{t-s} \left( S_s p_\varepsilon(\cdot - x) \int_0^s S_r G^\lambda(\cdot \cdot - x) dr \right) ds \right) \\ & \quad \times \left( \mu \left( \int_0^t S_s G^\lambda(\cdot \cdot - x) ds \right) \right)^{p-1} dx \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}^d} \mu \left( \int_0^t S_{t-s} \left( \int_0^s S_{s-r} \left( S_r p_\varepsilon(\cdot - x) \int_0^r S_u G^\lambda(\cdot - x) du \right) dr \right. \right. \\
& \quad \left. \left. \times \left( \int_0^s S_r G^\lambda(\cdot - x) dr \right)^{p-1} \right) ds \right) dx \\
& + \int_{\mathbb{R}^d} \mu \left( \int_0^t S_{t-s} \left( S_s p_\varepsilon(\cdot - x) \left( \int_0^s S_r G^\lambda(\cdot - x) dr \right)^p \right) ds \right) dx \\
& + \int_{\mathbb{R}^d} \mu \left( \int_0^t S_{t-s} \left( S_s p_\varepsilon(\cdot - x) \int_0^s S_{s-r} \left( \int_0^r S_u G^\lambda(\cdot - x) du \right)^{p'} dr \right) ds \right) dx \Big\} \\
& = c \sum_{i=1}^8 I_i(\varepsilon).
\end{aligned}$$

We will check the boundedness of all the terms  $I_i(\varepsilon)$ ,  $i = 1, \dots, 8$ . First note, that for  $d \leq \alpha$  all the terms  $I_i(\varepsilon)$ ,  $i = 1, \dots, 8$  can be bounded very easily, and we leave it to check to the reader. We will consider the case  $\alpha < d$ . The first two terms,  $I_1(\varepsilon)$  and  $I_2(\varepsilon)$  are easy to handle. By the Fubini theorem and Lemma 4 we get

$$I_1(\varepsilon) + I_2(\varepsilon) \leq \mu(1) (1 + (c_{4.19}t)^p).$$

By Lemma 7 we easily get

$$I_3(\varepsilon) + I_8(\varepsilon) \leq \mu(1)c(t).$$

For  $I_7(\varepsilon)$  we get the following

$$\begin{aligned}
I_7(\varepsilon) & = \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} p_{t-s}(y-z) p_{s+\varepsilon}(z-x) \\
& \quad \times \left( \int_0^s S_r G^\lambda(z-x) dr \right)^p ds dx dz \mu(dy) \\
& = \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} p_{t-s}(y-z) dz S_{s+\varepsilon} \left( \int_0^s S_r G^\lambda(\cdot) dr \right)^p (0) ds \mu(dy) \\
& = \mu(1) \int_0^t S_{s+\varepsilon} \left( \int_0^s S_r G^\lambda(\cdot) dr \right)^p (0) ds \\
& \leq \mu(1)c(t), \tag{4.29}
\end{aligned}$$

where the last inequality follows by Lemma 7. It is also easy to check that

$$\begin{aligned}
I_4(\varepsilon) & \leq \int_{\mathbb{R}^d} \mu \left( \int_0^t S_{t-s} \left( S_s p_\varepsilon(\cdot - x) \left( \left( \int_0^s S_r G^\lambda(\cdot - x) dr \right)^p + 1 \right) \right) ds \right) dx \\
& \leq I_7(\varepsilon) + \mu(1)t, \tag{4.30}
\end{aligned}$$

and

$$I_5(\varepsilon) \leq c_{4.19}^{p-1} I_4(\varepsilon).$$

The last term we have to handle is  $I_6(\varepsilon)$  :

$$\begin{aligned}
I_6(\varepsilon) &\leq \int_{\mathbb{R}^{3d}} \left( \int_{x:|y_1-x|\leq 1} + \int_{x:|y_1-x|>1} \right) \left( \int_0^t p_{t-s}(y-y_1) \right. \\
&\quad \times \int_0^s p_{s-r}(y_1-z) p_{r+\varepsilon}(z-x) \int_0^r S_u G^\lambda(z-x) du dr \\
&\quad \left. \times \left( \int_0^s S_r G^\lambda(y_1-x) dr \right)^{p-1} ds \right) dy_1 dz dx \mu(dy) \\
&= I_{6,1}(\varepsilon) + I_{6,2}(\varepsilon).
\end{aligned}$$

Our condition on  $d$  implies that we can choose  $\delta \in (0, 1/3)$  sufficiently small such that

$$(2\alpha - d)(p + 1) - \delta\alpha(p + 2) > -d. \quad (4.31)$$

By (4.20), Lemma 5 and Lemma 6 we get

$$\begin{aligned}
I_{6,1}(\varepsilon) &\leq c(t) \int_{\mathbb{R}^{3d}} \int_{x:|y_1-x|\leq 1} \left( \int_0^t p_{t-s}(y-y_1) \int_0^s p_{s-r}(y_1-z) (r+\varepsilon)^{\delta-1} \right. \\
&\quad \times (|z-x|^{3\alpha-2d-2\delta\alpha} \kappa(z-x) + |z-x|^{\alpha-d-\delta\alpha}) dr \\
&\quad \left. \times (|y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)} + 1) ds \right) dy_1 dz dx \mu(dy) \\
&= c(t) \int_{\mathbb{R}^{2d}} \int_{x:|y_1-x|\leq 1} \left( \int_0^t p_{t-s}(y-y_1) \int_0^s (r+\varepsilon)^{\delta-1} \right. \\
&\quad \times \int_{\mathbb{R}^d} p_{s-r}(y_1-z) (|z-x|^{3\alpha-2d-2\delta\alpha} \kappa(z-x) + |z-x|^{\alpha-d-\delta\alpha}) dz dr \\
&\quad \left. \times (|y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)} + 1) ds \right) dy_1 dx \mu(dy) \\
&\leq c(t) \int_{\mathbb{R}^{2d}} \int_{x:|y_1-x|\leq 1} \int_0^t p_{t-s}(y-y_1) \int_0^s (r+\varepsilon)^{\delta-1} \\
&\quad \times [1 + (s-r)^{\delta-1} (|y_1-x|^{4\alpha-2d-3\delta\alpha} + |y_1-x|^{2\alpha-d-2\delta\alpha} + 1)] dr \\
&\quad \times (|y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)} + 1) ds dy_1 dx \mu(dy) \\
&= c(t) \int_{\mathbb{R}^{2d}} \int_0^t p_{t-s}(y-y_1) \left[ \int_0^s (r+\varepsilon)^{\delta-1} dr \right. \\
&\quad \times \int_{x:|y_1-x|\leq 1} (1 + |y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)}) dx + \int_0^s (r+\varepsilon)^{\delta-1} (s-r)^{\delta-1} dr \\
&\quad \times \int_{x:|y_1-x|\leq 1} (1 + |y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)+4\alpha-2d-3\alpha\delta} \\
&\quad + |y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)+2\alpha-d-2\alpha\delta} + |y_1-x|^{(2\alpha-d-\delta\alpha)(p-1)} \\
&\quad \left. + |y_1-x|^{4\alpha-2d-3\alpha\delta} + |y_1-x|^{2\alpha-d-2\alpha\delta}) dx \right] dy_1 \mu(dy) \\
&\leq c(t) \mu(1),
\end{aligned}$$

where the last inequality follows by (4.31). As for the  $I_{6,2}(\varepsilon)$ , by Lemma 6 we get

$$\begin{aligned} I_{6,2}(\varepsilon) &\leq \sup_{x, y_1: |y_1 - x| > 1} \left( \int_0^t S_r G^\lambda(y_1 - x) dr \right)^{p-1} \int_{\mathbb{R}^{3d}} \int_0^t p_{t-s}(y - y_1) \\ &\quad \times \int_0^s \int_0^r \int_0^\infty p_{s-r}(y_1 - z) p_{r+\varepsilon+u+v}(0) e^{-\lambda v} dv du dr ds dy_1 dz \mu(dy) \\ &\leq c(t) \mu(1) \int_0^r \int_0^s \int_0^r \int_0^\infty (r + \varepsilon + u + v)^{-d/\alpha} e^{-\lambda v} dv du dr ds \end{aligned}$$

and the last integral is bounded if  $d < 3\alpha$ . By combining all the above estimates we are done with the first part of the proposition.

Now we are going to prove the second part of the proposition. Take

$$\varphi(y) = \left( \int_0^t \int G^{\lambda, \varepsilon}(y - x) Y_s^K(dx) ds \right)^p, \quad y \in \mathbb{R}^d.$$

For each  $n \in \mathbb{N}$  define the truncations functions,  $\varphi_n = \varphi \wedge n$ . Then  $0 \leq \varphi_n \uparrow \varphi$ , as  $n \rightarrow \infty$ . Since  $\varphi_n$  is bounded and  $p_1(z) dz Y_t^K(dy)$  is a finite measure, we have by the dominated convergence theorem and the scaling relationship (4.23) the following estimation

$$\begin{aligned} \int \varphi_n(x) Y_t^K(dx) &= \lim_{\delta \downarrow 0} \int \int \varphi_n(\delta^{1/\alpha} z + x) p_1(z) dz Y_t^K(dx) \\ &\leq \liminf_{\delta \downarrow 0} \int \int \varphi(\delta^{1/\alpha} z + x) p_1(z) dz Y_t^K(dx) \\ &= \liminf_{\delta \downarrow 0} \int \int p_\delta(x - y) Y_t^K(dx) \varphi(y) dy. \end{aligned}$$

Letting  $n \rightarrow \infty$ , by the monotone convergence theorem, we have

$$\begin{aligned} &\int \left( \int_0^t Y_s^K(G^{\lambda, \varepsilon}(\cdot \cdot - x)) ds \right)^p Y_t^K(dx) \\ &\leq \liminf_{\delta \downarrow 0} \int \left( \int_0^t Y_s^K(G^{\lambda, \varepsilon}(\cdot \cdot - x)) ds \right)^p Y_t^K(p_\delta(\cdot - x)) dx. \end{aligned}$$

From the Fatou lemma we get

$$\begin{aligned} &E \left[ \int \left( \int_0^t Y_s^K(G^{\lambda, \varepsilon}(\cdot \cdot - x)) ds \right)^p Y_t^K(dx) \right] \\ &\leq \liminf_{\delta \downarrow 0} E \left[ \int \left( \int_0^t Y_s^K(G^{\lambda, \varepsilon}(\cdot \cdot - x)) ds \right)^p Y_t^K(p_\delta(\cdot - x)) dx \right]. \end{aligned}$$

Since  $G^{\lambda, \varepsilon} \leq G^\lambda$  we have by the already proven part of Proposition 2,

$$\begin{aligned} &E \left[ \int \left( \int_0^t Y_s^K(G^{\lambda, \varepsilon}(\cdot \cdot - x)) ds \right)^p Y_t^K(dx) \right] \\ &\leq \liminf_{\delta \downarrow 0} E \left[ \int \left( \int_0^t Y_s^K(G^\lambda(\cdot \cdot - x)) ds \right)^p Y_t^K(p_\delta(\cdot - x)) dx \right] \\ &\leq c(K, p, d, \alpha, \beta). \end{aligned}$$

Using once again the monotone convergence theorem, as  $\varepsilon \rightarrow 0$ , we arrive at

$$E \left[ \int \left( \int_0^t Y_s^K(G^\lambda(\cdot - x)) ds \right)^p Y_t^K(dx) \right] < c(K, p, d, \alpha, \beta),$$

and we are done.  $\square$

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