Non-cooperative Game Theory

Part II:

Games in Informational Form

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This is not a final version. Comments are welcome.
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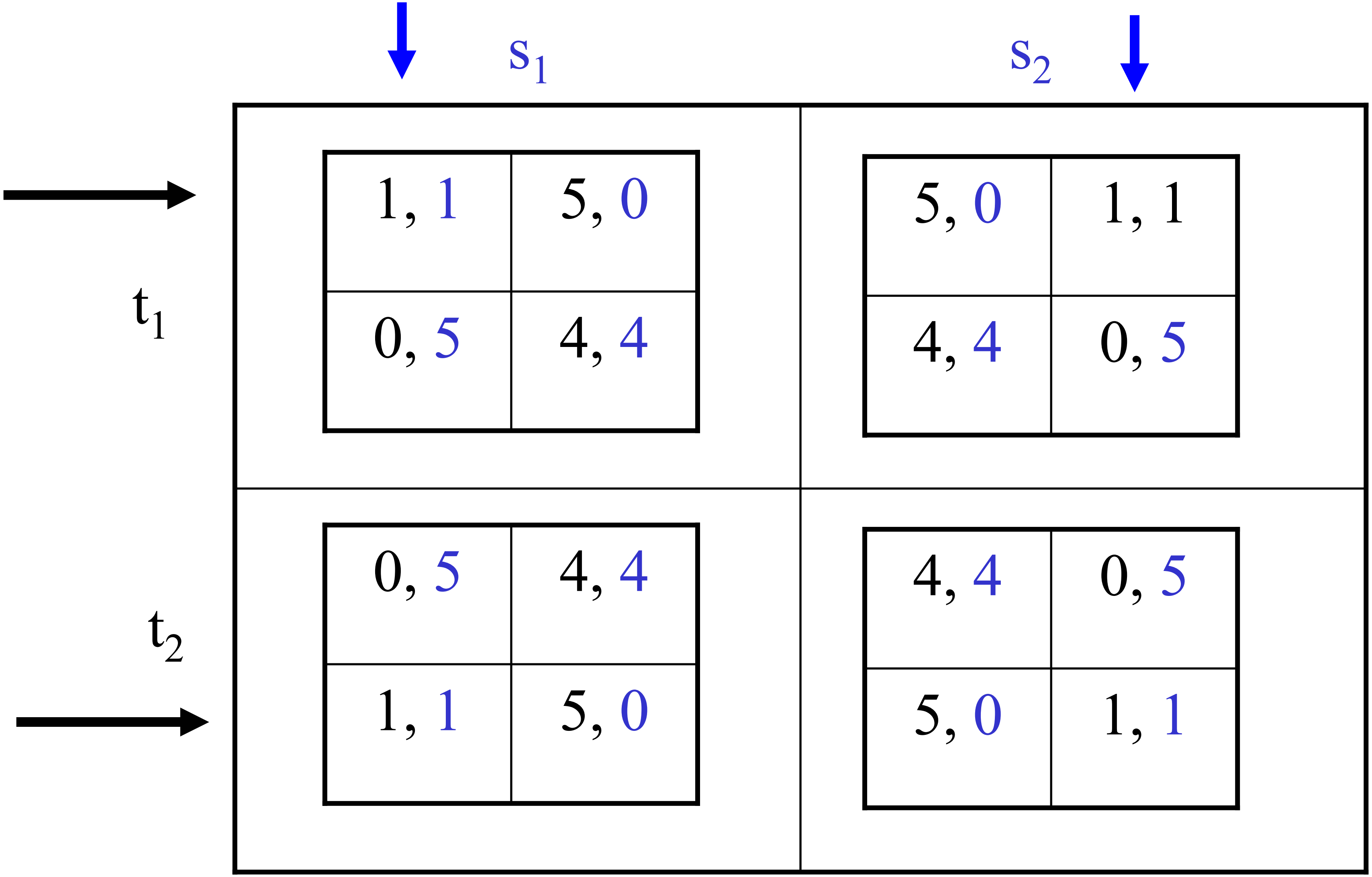
An Example (Example 1):

	s_1		s_2	
t_1	a_{1111} b_{1111}	a_{1112} b_{1112}	a_{1211} b_{1211}	a_{1212} b_{1212}
	a_{1121} b_{1121}	a_{1122} b_{1122}	a_{1221} b_{1121}	a_{1222} b_{1222}
t_2	a_{2111} b_{2111}	a_{2112} b_{2112}	a_{2211} b_{2211}	a_{2212} b_{2212}
	a_{2121} b_{2121}	a_{2122} b_{2122}	a_{2221} b_{2221}	a_{2222} b_{2222}

Games with Probabilistic Information (Bayesian Games)

	s_1		s_2	
t_1	a_{1111} b_{1111}	a_{1112} b_{1112}	a_{1211} b_{1211}	a_{1212} b_{1212}
	a_{1121} b_{1121}	a_{1122} b_{1122}	a_{1221} b_{1121}	a_{1222} b_{1222}
		p_{11}		p_{12}
t_2	a_{2111} b_{2111}	a_{2112} b_{2112}	a_{2211} b_{2211}	a_{2212} b_{2212}
	a_{2121} b_{2121}	a_{2122} b_{2122}	a_{2221} b_{2221}	a_{2222} b_{2222}
		p_{21}		p_{22}

Domination in Games in IF



<table>
<tr><th></th><th>s₁</th><th>s₂</th></tr>
<tr><td>t₁</td>
<td><table><tr><td>1, 1</td><td>5, 0</td></tr><tr><td>0, 5</td><td>4, 4</td></tr></table></td>
<td><table><tr><td>5, 0</td><td>1, 1</td></tr><tr><td>4, 4</td><td>0, 5</td></tr></table></td></tr>
<tr><td>t₂</td>
<td><table><tr><td>0, 5</td><td>4, 4</td></tr><tr><td>1, 1</td><td>5, 0</td></tr></table></td>
<td><table><tr><td>4, 4</td><td>0, 5</td></tr><tr><td>5, 0</td><td>1, 1</td></tr></table></td></tr>
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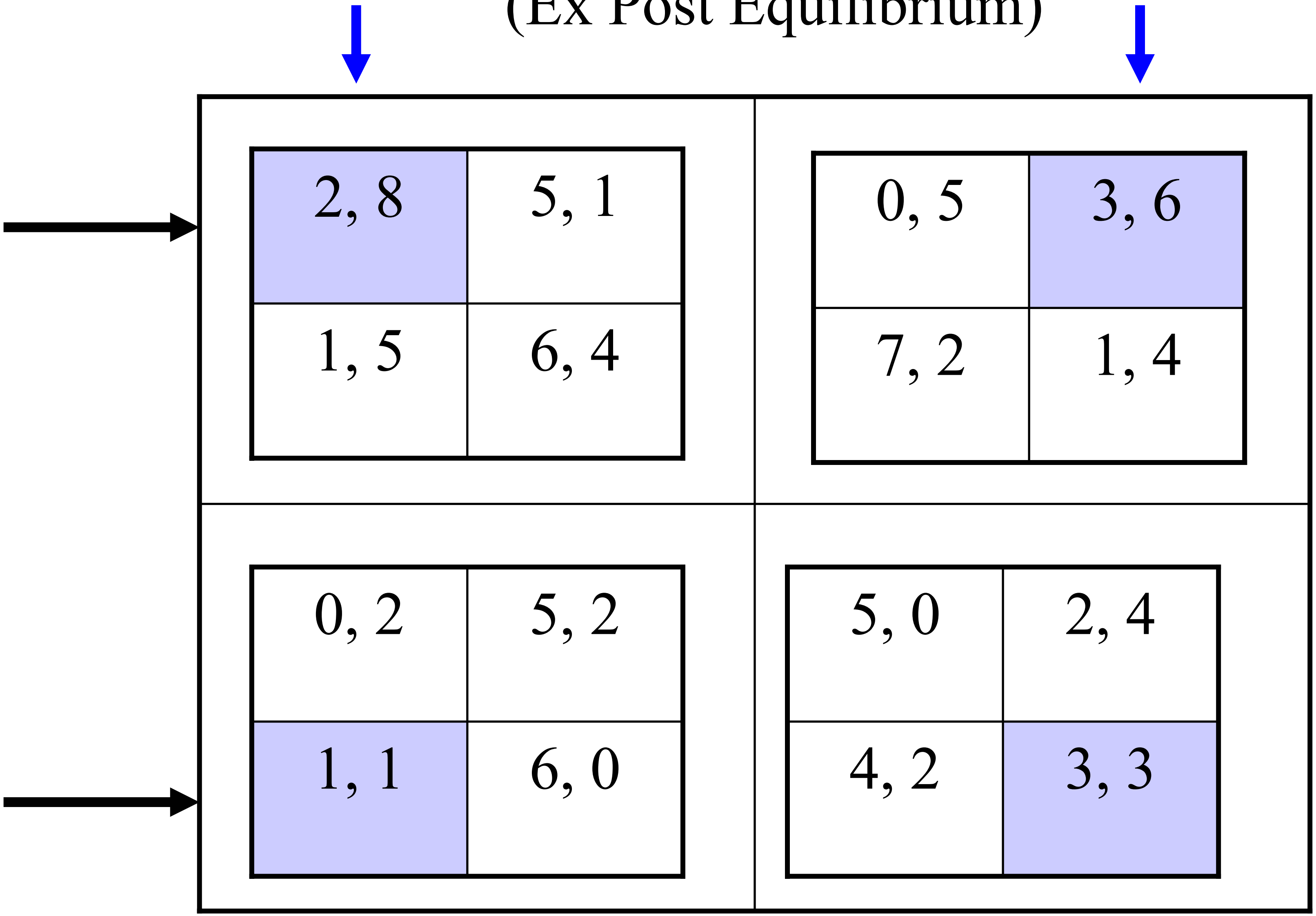
Equilibrium in Games in IF
(Ex Post Equilibrium)



2, 8	5, 1
1, 5	6, 4

0, 5	3, 6
7, 2	1, 4

0, 2	5, 2
1, 1	6, 0

5, 0	2, 4
4, 2	3, 3

Formal Definitions

PDF

1 Ex post equilibrium in games in informational form

A game in **informational form** $G = G(N, \Omega, T, (\tilde{t}_i)_{i \in N}, X, (u_i)_{i \in N})$ is defined by the following parameters:

- **Agents**: Let $N = \{1, \ldots, n\}$ be the set of agents.
- **States**: Let Ω be the set of (relevant) states.
- **Types**: Let T_i be the set of types of agent i, $T = \times_{i \in N} T_i$.
- **Signaling functions**: Let $\tilde{t}_i : \Omega \to T_i$ be the signaling function of agent i.

Without loss of generality, it is assumed that every type $t_i \in T_i$ is possible. That is $\tilde{t}_i(\Omega) = T_i$.

- **Actions**: Let X_i be the set of actions of i, $X = \times_{i \in N} X_i$.
- **Utility functions**: Let $u_i(\omega, x)$ be the utility of i at state ω, when the agents choose the action profile x.

Every $\omega \in \Omega$ defines a game in strategic (normal) form, $G(\omega)$. In this game agent i receives $u_i(\omega, x)$, when it chooses x_i, and all other agents choose x_{-i}. However, the agents do not know which game they play.

For $t_i \in T_i$ let $\Omega_i(t_i)$ be the set of states that generate the signal t_i, that is

$$\Omega_i(t_i) = \{\omega \in \Omega | \tilde{t}_i(\omega) = t_i\}.$$

A game in informational form is **finite** if the set of states and the strategy sets are finite.

A **strategy** of i is a function $b_i : T_i \to X_i$; The associated **implied strategy** is the function $\hat{b}_i : \Omega \to X_i$ given by

$$\hat{b}_i(\omega) = b_i(\tilde{t}_i(\omega)).$$

A strategy b_i of i is an **ex post dominant strategy** for i, if for every profile of strategies b_{-i} of the other players, for every $t_i \in T_i$, for every

$\omega \in \Omega_i(t_i)$, and for every $x_i \in X_i$,

$$u_i(\omega, b_i(t_i), \hat{b}_{-i}(\omega)) > u_i(\omega, x_i, \hat{b}_{-i}(\omega)).$$

It is **weak** ex post dominant strategy for i if the above inequalities are weak.

Note that we do not require that a strict inequality holds at least in one of these inequalities. This is one of many inconsistencies , which are common in the literature of game theory.

A profile of strategies $b = (b_1, \cdots, b_n)$ is an **ex post equilibrium**, if for every agent i, for every $t_i \in T_i$, for every $\omega \in \Omega_i(t_i)$, and for every $x_i \in X_i$,

$$u_i(\omega, b_i(t_i), \hat{b}_{-i}(\omega)) \geq u_i(\omega, x_i, \hat{b}_{-i}(\omega)).$$

Obviously, if b_i is an ex post dominant strategy for every i, b is an ex post equilibrium, but not necessarily vice versa.

An ex post equilibrium b, in which every strategy b_i is ex post dominant is called an ex post **domination equilibrium**.

An Example: Mechanism Design

A (good) government must choose one of m alternatives, out of the set of alternatives $A=\{a_1,a_2,\ldots,a_m\}$. All alternatives have the same cost. Every agent i in this society, i in $N=\{1,2,3,\ldots,n\}$ is ready to pay $v_i(a_j)$ for alternative a_j. However, the vector of i's valuations is i's <u>private information.</u> The goal of the government is to choose an <u>optimal alternative</u> , that is an alternative that maximizes the total social surplus, $\sum_{\{i \text{ in } N\}} v_i(a_j)$ over a_j in A.

The government can ask every agent to submit her valuation vector, but some agents may not be truthful. The government can use the following mechanism:

The VCG Mechanism

(V=Vickrey, C=Clarke, G=Groves)

- Every agent i is required to submit a valuation vector $w_i=(w_i(a_1),w_i(a_2),\ldots,w_i(a_m))$.

- The government chooses an optimal alternative , $a= a(w_1,w_2,\ldots,w_n)$ with respect to the reported valuations.

- Every agent i pays the difference $c_i= c_i(w_1,w_2,\ldots,w_n)$ between the social surplus of the society if she does not exist, and the social surplus of the rest of the agents when she is present. that is:

$$\boxed{c_i=\mathrm{Max}_{\{1\leq j\leq m\}} \sum_{\{k \text{ in } N\setminus i\}} w_k(a_j) -\sum_{\{k \text{ in } N\setminus i\}} w_k(a).}$$

II

Note that the government created a game in informational form in which, the set of players is N. The set of states is $\Omega=(R_+^m)^n$, the signaling function of i is $t_i^{\sim}(v_1,v_2,\ldots,v_n)=v_i$, the space of messages is $M=R_+^m$, and the utility function of i is

$$\boxed{u_i(v,w)=v_i(a(w))-c_i(w),}$$

where $v=(v_i,v_{-i})$, and $w=(w_i,w_{-i})$.

Theorem 1: Truth telling is a dominant strategy for every agent i in the game in IF generated by the VCG mechanism.

III

<u>Proof</u>: Let i be an agent with the valuation function i. We have to prove that for every w_i, for every v_{-i}, and for every w_{-i},

$u_i(v_i,v_{-i},v_i,w_{-i}) \geq u_i(v_i,v_{-i},w_i,w_{-i})$.

That is, we have to show that

$v_i(a(v_i,w_{-i})-c_i(v_i,w_{-i}) \geq v_i(a(w_i,w_{-i})-c_i(w_i,w_{-i})$.

Plugging in the formula for c_i, we have to show that

$v_i(a)+\sum_{\{k\neq i\}} v_k(a) \geq v_i(a')+\sum_{\{k\neq i\}} v_k(a')$,

where $a=a(v_i,w_{-i})$, and $a'=a(w_i,w_{-i})$.

Because the alternative a is chosen to maximize the left-hand-side of the above inequality, the proof is completed. QED

IV

Note that when the VCG mechanism is applied, every agent reports her true valuation vector because it is optimal for her to do it. Consequently, the government chooses a socially optimal alternative. Moreover, the mechanism is fair in the sense that every agent i pays less than her willingness to pay for the chosen alternative!!! That is,

$$\boxed{v_i(a) \leq c_i.}$$

The VCG mechanism is a marvelous mechanism from Economics point of view, but it is a nightmare for software engineers. The computational problems of finding an optimal allocation and the associated payments are NP complete.

(One way for) getting 100 as your course's grade

You will get 100 in this course if you find a mechanism in which, every student has a unique dominant strategy, and such that when everyone acts optimally, every student receives the grade he believes he deserves.

Very Simple Auctions

In a <u>first-price</u> auction, the winner is the bidder with the highest bid. She pays her bid.
In case of a tie, the winner is the bidder with the lowest index (amongst the bidders that submit the highest bid). Assume every bidder i has a valuation v_i, which reflects her maximal willingness to pay, and which is her private value.
<u>Exercise</u>: Formulate the game in IF generated by a first-price auction, and show that the strategy of bidding 90% of your valuation is not dominant.

II

In a second-price auction the winner is the bidder with the highest bid, and she pays the highest non-winning bid. Ties are treated as in first-price auctions. For example, if the bids are $b_1=100$, $b_2=110$, $b_3=90$, agent 2 wins the object, and she pays 100.

If the bids are $b_1=110$, $b_2=110$, $b_3=90$, agent 1 wins the object and he pays 110.

Assume every bidder i

has a valuation v_i, which reflects her maximal willingness to pay, and which is her private value.

A second-price auction is called a Vickrey Auction

III

Exercise: Formulate the game in IF generated by a second-price auction, and show that the strategy of bidding the true valuation is dominant!

Mixed Strategies and Behavioral Strategies

Consider the game in IF in Slide 2 (Example 1). Player 1 has 4 strategies (she can go up or down given any signal, t_1, or t_2). Let us denote these strategies by b_l, l=1,2,3,4. More precisely:

	t_1	t_2
b_1	U	U
b_2	U	D
b_3	D	U
b_4	D	D

II

Therefore, a mixed strategy of player 1 is a probability vector $p=(p_1,p_2,p_3,p_4)$. The interpretation is as follows. Before player 1 receives a signal t_1 or t_2, she conducts a lottery to choose her strategy. The probability to choose b_1 is p_1. However, player 1 can conduct a lottery after she receives her signal. If she does so, we say that she is using a behavioral strategy. In this example a behavioral strategy of player 1 is described by a vector $((a_1,a_2),(b_1,b_2))$ of probability vectors. The interpretation is as follows:

If the signal is t_1, player 1 chooses U with prob. a_1, and D with probability a_2.

If the signal is t_2, player 1 chooses U with prob. b_1, and D with probability b_2.

III

Note that the dimension of the set of mixed strategies is 3 (player 1 has to choose 3 independent numbers, and the forth one is determined by these three numbers because $p_1+p_2+p_3+p_4=1$), while the dimension of the set of behavioral strategies is 2 only.

Definition: A behavioral strategy for i is a function $f_i: S_i \rightarrow \Lambda(X_i)$, where S_i is the set of signals of i, and $\Lambda(X_i)$ is the set of probability vectors over X_i.

Note that if the number of actions of i is k and the number of signals of i is m, then the dimension of the set of mixed strategies is k^m -1, while the dimension of the set of behavioral strategies is m(k-1).

IV

Obviously $m(k-1) < k^m - 1$.

For example, when $k=2$, and $m=7$, the dimension of the set of behavioral strategies is less than 1% from the dimension of the set of mixed strategies! Therefore, the following theorem is important:

<u>Theorem 2</u>: The set of behavioral strategies and the set of mixed strategies are equivalent.

The definition of equivalent strategies, as well as the precise meaning of Theorem 2 will be given at class. However, because of this theorem we rarely deal with mixed strategies in games with IF. We restrict attention to behavioral strategies.

Ex Post Equilibrium with Behavioral Strategies

		s_1		s_2	
		0.5	**0.5**	**0.5**	**0.5**
t_1	**0.5**	1, -1	0, 0	2, -2	0, 0
	0.5	0, 0	1, -1	0, 0	2, -2
t_2	**0.75**	1, -1	0, 0	2, -2	0, 0
	0.25	0, 0	1, -3	0, 0	2, -6