MATHEMATICS FOR NON-COOPERATIVE GAME THEORY[†]

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1.Compact Sets. In this course we assume that all (infinite) strategy sets are subsets of some Euclidean set. That is $X_i \subseteq R^m$ for some integer $m \geq 1$ (m may depend on i). That is $x_i \in X_i$ means $x_i = (x_{i_1}, x_{i_2}, ..., x_{i_m})$. Thus if $x_i \in X_i \subseteq R^{m_i}$, then every $x = (x_1, x_2, ...x_n) \in X = X_1 \times X_2 \times \cdots \times X_n$ is a vector in $R^{m_1 + m_2 \cdots + m_n}$. That is

$$x = (x_{1_1}, x_{1_2}, ..., x_{1_{m_1}}, x_{2_1}, ..., x_{2_{m_2}}, ..., x_{n_{m_n}}).$$

Recall that for $x \in R^m$, the **norm** of x is the distance between x and 0 (where $0 = (0, 0, ..., o)$. That is

$$||x|| = (\sum_{j=1}^{m} x_j^2)^{0.5}.$$

The distance between x and z in R^m is $||x - z||$. Recall the following properties of the norm:

(1) $||\alpha X|| = |\alpha| ||X||$ for all $x \in R^m$ and $\alpha \in R$.
(2) $||x + y|| \leq ||x|| + ||y||$.
(3) $||x|| = 0$ if and only if $x = 0$.

We say that $\lim_{n \to \infty} x^n = x$ if the sequence of numbers $(||x^n - x||)_{n=1}^{\infty}$ is converging to zero. That is, the distance between x^n and x is converging to zero.

A subset $K \subseteq R^m$ is **closed** if it has the following property:

If $x_n \in K$ for every $n \geq 1$, and $\lim_{n \to \infty} x^n = x$, then $x \in X$. That is, K is closed if every limit point of vectors in K belongs to K.

$K \subseteq R^m$ is **bounded**, if there exists a number $C > 0$ such that $||x|| \leq C$ for every $x \in K$.

K is **compact** if it is both, closed and bounded.

Lemma m0. *If $K_i \subseteq R^{m_i}$ is a compact set for all $1 \leq i \leq n$, then $K = K_1 \times K_2 \times \cdots \times K_n$ is also a compact set.*

[†] (c) Dov Monderer. dov@ie.technion.ac.il .

2. Continuous Functions. Let $A \subseteq R^m$ and let f be a function defined on A and has values in R^s, that is $f : A \to R^s$. f is **continuous** if for every converging sequence $\lim_{n\to\infty} x^n = x$ such that x_n, $n \geq 1$, and x belong to A, $\lim_{n\to\infty} f(x_n) = f(x)$.

Equivalently, f is continuous if for every $x^0 \in A$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in A$ with $||x - x^0|| < \delta$, $||f(x) - f(x^0)|| < \varepsilon$.

Theorem m1. *Let $f : K \to R^s$ be a continuous function, where K is compact. Then f is uniformly continuous. That is, for every $\varepsilon > 0$ there exists $\delta > 0$, such that for every $x, y \in K$ with $||x - y|| < \delta$, $||f(x) - f(y)|| < \varepsilon$.*

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Theorem m2. *Let $f : K \to R$ be a continuous function. Then f achieves its maximal and minimal values in K. That is, there exist $x_m, x_M \in K$ such that*

$$f(x_m) \leq f(x) \leq f(x_M) \quad \textit{for every } x \in K.$$

3.Convex Sets. A subset $A \subseteq R^m$ is **convex** if for any $x, y \in A$, the line that joins x with y lies in A. That is, for every $0 \leq \alpha \leq 1$, $\alpha x + (1 - \alpha)y \in A$. Let $x, x^1, ..., x^k$, $k \geq 1$ in R^m. x is a **convex combination** of $\{x^1, x^2, \ldots, x^k\}$, if there exist non-negative real numbers $\alpha_1, \ldots, \alpha_k$ with $\sum_{j=1}^k \alpha_j = 1$, such that

$$x = \sum_{j=1}^{k} \alpha_j x^j.$$

Theorem m3. *A is convex if and only if every convex combinations of points in A belongs to A.*

Note that the empty set $\emptyset$ and the whole space R^m are convex sets. Let $B \subseteq R^m$ (possibly finite). The set of all convex combinations of points in B is called the **convex hull** of B and it is denoted by $co(B)$.

Theorem m4. *$co(B)$ is a convex set that contains B. Moreover, $co(B)$ is the smallest convex set that contains B. That is, if K is convex and $B \subseteq K$, $co(B) \subseteq K$.*

. Let A be a convex set, and let $x \in A$. x is an **extreme point of** A, if x does not belong to the interior of any segment in A. That is, for every $z, y \in A$, $z \neq y$, and for every $0 < \alpha < 1$,

$$x \neq \alpha z + (1 - \alpha)y.$$

Theorem m5. *Every compact and convex set is the convex hull of its extreme points. That is*

$$K = co(E(K)),$$

where $E(K)$ *is the set of extreme points of* K*.*

And..

Theorem m6. *Let* B *be a finite set of points in* R^m*. Then,* $co(B)$ *is a compact and convex set. Moreover, every extreme point of* $co(B)$ *belongs to* B*, that is* $E(co(B)) \subseteq B$*.*

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