Service Engineering

Phase-Type Distributions

City-Hall clerk: Phase-Type Service-Time in Administrative Queues.
A city council clerk is responsible for handling incoming mail. One can represent the arrival of documents to the clerk by a Poisson process with the constant arrival rate of 5 documents per hour. We can divide the handling time of a document into the two stages: *reading time* and *processing time*. The average reading time is equal to 3 minutes. It was found that 30% of the documents require *long processing time* (5 minutes, on average), 60% require *short processing time* (2 minutes, on average) and the rest of the documents are sent to a waste basket immediately after reading. If two documents or more are on the clerk's desk (including the document he is working on), any new arriving documents are sent to his assistant.

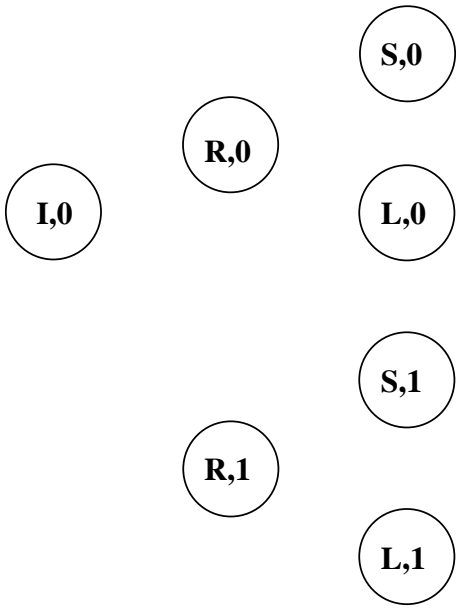
In order to describe the activities of the clerk, define a stochastic process
$X = \{X(t),\ t \geq 0\}$ by

$$X(t) = (X_1(t), X_2(t)),\ t \geq 0,$$

where
$X_1(t) = I$, if the server is **I**dle at time t;
$X_1(t) = R$, if the server is **R**eading a document at time t;
$X_1(t) = L$, if the server is engaged in a **L**ong processing at t;
$X_1(t) = S$, if the server is engaged in a **S**hort processing at t;
$X_2(t) = 1$, if there is a document awaiting for treatment at time t;
$X_2(t) = 0$, if there is no document in the queue at t.
Under certain assumptions, it is feasible to represent $X = \{X(t),\ t \geq 0\}$ as a Markov Jump Process on the state space

$$S\ =\ \{(I,0),(R,0),(R,1),(S,0),(S,1),(L,0),(L,1)\}.$$

As usual, one can describe the process using a transition-rates diagram that depicts the process states via the following graph nodes:



1. Describe the service time distribution, as accurately as possible. Calculate its expected value (average handling time of a document).

2. Calculate the standard deviation of the service time distribution, as described in Question 5.

3. Formulate the assumptions that would imply the Markov property of the process X.

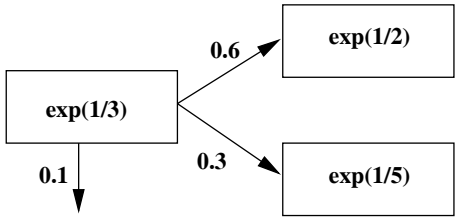
4. Complete the transitions-rate diagram, adding appropriate directed arcs and transition rates above the arcs.

5. Solve the steady-state equations.

6. Compute the fraction of documents handled by the clerk's assistant.

City-Hall clerk: Phase-Type Service-Time in Administrative Queues. Solution.

1. The service time distribution is *phase-type*:



$$\text{E[service time]} \;=\; 3 + 0.6 \cdot 2 + 0.3 \cdot 5 \;=\; 5.7 \text{ min.}$$

2. Reading time is independent of processing time, therefore

$$\text{Var[service time]} \;=\; \text{Var[reading time]} + \text{Var[processing time]},$$

$$\text{Var[reading time]} \;=\; \frac{1}{(1/3)^2} \;=\; 9\ .$$

The processing time X can be represented in the following form:

$$X \;=\; \begin{cases} \exp\left(\frac{1}{5}\right)\ , & p = 0.3 \\ \exp\left(\frac{1}{2}\right)\ , & p = 0.6 \\ 0\ , & p = 0.1 \end{cases}$$

Note that the second moment of an exponential random variable $Y \sim \exp(\lambda)$ is given by:

$$\text{E}[Y^2] \;=\; (\text{E}Y)^2 + \text{Var}[Y] \;=\; \frac{1}{\lambda^2} + \frac{1}{\lambda^2} \;=\; \frac{2}{\lambda^2}$$

Then

$$\text{E}[X^2] \;=\; 0.3 \cdot \text{E}[\{\exp\left(\frac{1}{5}\right)\}^2] + 0.6 \cdot \text{E}[\{\exp\left(\frac{1}{2}\right)\}^2] \;=\; 0.3 \cdot 50 + 0.6 \cdot 8 \;=\; 19.8$$

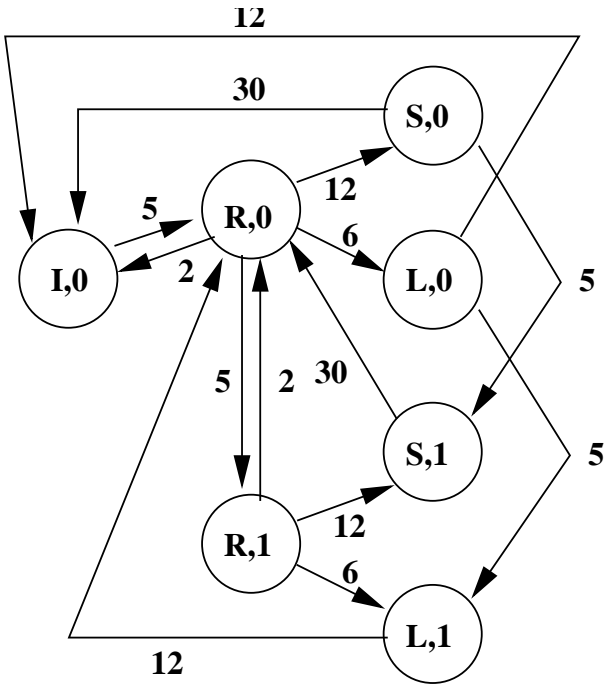
$$\text{Var}[X] \;=\; \text{E}[X^2] - (\text{E}X)^2 \;=\; 12.51$$

$$\text{Var[service time]} \;=\; 9 + 12.51 \;=\; 21.51$$

The standard deviation of the service time distribution is equal to $\sqrt{21.51} = 4.64$.

3. Reading times, short processing times and long processing times are *exponentially distributed*. The stochastic components of the system (interarrival times, service times, switches between states) are *independent*.

4.



5.

$$\begin{cases} 5\pi_{I0} = 2\pi_{R0} + 30\pi_{S0} + 12\pi_{L0} \\ 20\pi_{R1} = 5\pi_{R0} \\ 35\pi_{S0} = 12\pi_{R0} \\ 17\pi_{L0} = 6\pi_{R0} \\ 30\pi_{S1} = 12\pi_{R1} + 5\pi_{S0} \\ 12\pi_{L1} = 6\pi_{R1} + 5\pi_{L0} \\ \pi_{I0} + \pi_{R0} + \pi_{S0} + \pi_{L0} + \pi_{R1} + \pi_{S1} + \pi_{L1} = 1 \end{cases} \Longrightarrow \begin{cases} \pi_{I0} = 0.582 \\ \pi_{R0} = 0.176 \\ \pi_{S0} = 0.060 \\ \pi_{L0} = 0.062 \\ \pi_{R1} = 0.044 \\ \pi_{S1} = 0.028 \\ \pi_{L1} = 0.048 \end{cases}$$

6. The clerk's assistant handles 12% of the documents since

$$\pi_{R1} + \pi_{S1} + \pi_{L1} \;=\; 0.12 \;\; \text{(PASTA)}$$