We provide epistemic conditions for Nash equilibrium, which are considerably weaker than the standard ones by Aumann and Brandenburger (1995). Indeed, we simultaneously replace common belief in conjectures, mutual belief in rationality and mutual belief in payoffs with strictly weaker epistemic conditions of pairwise common belief in conjectures, pairwise mutual belief in rationality and pairwise mutual belief in payoffs, respectively. It is also shown that, unlike Aumann and Brandenburger's conditions, ours do not imply common belief in rationality in complete information games. Surprisingly, they actually do not even imply mutual belief in rationality.

Keywords: Nash equilibrium, pairwise common belief, pairwise mutual belief, rationality, conjectures, epistemic game theory.

1. INTRODUCTION

In their seminal paper, Aumann and Brandenburger (1995) provided epistemic conditions for Nash equilibrium. Accordingly, if there exists a common prior, then mutual belief in rationality and payoffs as well as common belief in each player’s conjecture about the opponents’ strategies imply Nash equilibrium in normal form games with more than two players. As they pointed out, in their epistemic conditions common knowledge enters the picture in an unexpected way; in fact, they stressed that what is needed is common knowledge of the players’ conjectures and not of the players’ rationality (Aumann and Brandenburger, 1995, p. 1163). Their result challenged the widespread view that common belief in rationality is essential for Nash equilibrium. Subsequently, Polak (1999) showed that in complete information games, Aumann and Brandenburger’s conditions actually do imply common belief in rationality. In a sense, his result thus restored some of the initial confidence in the importance of common belief in rationality for Nash equilibrium.

Here, we introduce weaker epistemic conditions for Nash equilibrium than those by Aumann and Brandenburger (1995), by simultaneously relaxing their main assumptions. Our new conditions are based on imposing pairwise mutual belief in rationality, pairwise mutual belief in payoffs and pairwise common belief in conjectures only for some pairs of players. This constitutes a significant weakening of Aumann and Brandenburger’s epistemic foundation, as their conditions correspond to pairwise mutual belief...
belief in rationality, pairwise mutual belief in payoffs and pairwise common belief in conjectures for all pairs of players. For instance, such conditions on pairwise interactive belief could emerge as the steady state of a sequence of pairwise communication correspondences between connected agents in a network. Note that this difference is particularly important for large games, such as economies with many agents.

Apart from generalizing Aumann and Brandenburger’s standard result, we also contribute to the debate about the connection between common belief in rationality and Nash equilibrium. Indeed, we prove that – contrary to what Polak (1999) showed for Aumann and Brandenburger’s foundation – our conditions do not entail common belief in rationality. Surprisingly, they actually do not even imply mutual belief in rationality. Thus, Aumann and Brandenburger’s intuition about common belief in rationality not being essential for Nash equilibrium is confirmed.

2. PRELIMINARIES

2.1. Games

Let \((I, (A_i)_{i \in I}, (g_i)_{i \in I})\) be game in normal form, where \(I = \{1, \ldots, n\}\) denotes the finite set of players with typical element \(i\), and \(A_i\) denotes the finite set of strategies, also called actions, with typical element \(a_i\) for every player \(i \in I\). Moreover, define \(A := \times_{i \in I} A_i\) with typical element \(a = (a_1, \ldots, a_n)\) and \(A_{-i} := \times_{j \in I \setminus \{i\}} A_j\) with typical element \(a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)\). The function \(g_i : A_i \times A_{-i} \rightarrow \mathbb{R}\) denotes player \(i\)’s payoff function.

A probability measure \(\phi_i \in \Delta(A_{-i})\) on the set of the opponents’ action combinations is called a conjecture of \(i\), with \(\phi_i(a_{-i})\) signifying the probability that \(i\) attributes to the opponents playing \(a_{-i}\). Slightly abusing notation, let \(\phi_i(a_j) := \text{marg}_{A_j} \phi_i(a_j)\) denote the probability that \(i\) assigns to \(j\) playing \(a_j\). Note that it is standard to admit correlated beliefs, i.e. \(\phi_i\) is not necessarily a product measure, hence the probability \(\phi_i(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)\) can differ from the product \(\phi_i(a_1) \cdots \phi_i(a_{i-1}) \phi_i(a_{i+1}) \cdots \phi_i(a_N)\) of the marginal probabilities.\(^2\) We say that an action \(a_i\) is a best response to \(\phi_i\), and write \(a_i \in BR_i(\phi_i)\), whenever

\[
\sum_{a_{-i} \in A_{-i}} \phi_i(a_{-i}) g_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in A_{-i}} \phi_i(a_{-i}) g_i(a'_i, a_{-i})
\]

for all \(a'_i \in A_i\).

A randomization over a player’s actions is called mixed strategy, and is typically denoted by \(\sigma_i \in \Delta(A_i)\) for all \(i \in I\). Let \(\Delta(A_1) \times \cdots \times \Delta(A_n)\) denote the space of mixed strategy profiles, with typical element \((\sigma_1, \ldots, \sigma_n)\). Slightly abusing terminology, we say that a pure strategy \(a_i \in A_i\) is a best response to

\(^2\) Intuitively, a player’s belief on his opponents’ choices can be correlated, even though players choose independently from each other.
\( \sigma \), and write \( a_i \in BR_i(\sigma) \), whenever \( a_i \) is a best response to the product measure \( \sigma_{-i} := \text{marg}_{A_{-i}} \sigma \), which is an element of \( \Delta(A_1) \times \cdots \times \Delta(A_{i-1}) \times \Delta(A_{i+1}) \times \cdots \times \Delta(A_n) \). Nash’s notion of equilibrium can then be defined as follows: a mixed strategy profile \((\sigma_1, \ldots, \sigma_n)\) is a Nash equilibrium of the game \( \Gamma \), whenever \( a_i \in BR_i(\sigma) \) for all \( a_i \in \text{supp}(\sigma_i) \) and for all \( i \in I \).

### 2.2. Epistemic Models

The epistemic approach to game theory analyzes the relation between knowledge, belief, and actions of rational players. While classical game theory is based on the two basic primitives – game form and actions – epistemic game theory adds an epistemic framework as a third elementary component so that knowledge and beliefs can be explicitly modeled in games.

Following Aumann and Brandenburger (1995), let \( S_i \) be a finite set of types\(^3\) for each player \( i \), with typical element \( s_i \). As usual, let \( S := S_1 \times \cdots \times S_n \) and \( S_{-i} := S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n \). An element \( s = (s_1, \ldots, s_n) \) of \( S \) is called state of the world, or simply state, while every subset of \( S \) is called an event. The event \( [s_i] := \{ s \in S : \text{proj}_{S_i} s = s_i \} \) contains all states at which \( i \)'s type is \( s_i \).

Each type \( s_i \in S_i \) is associated with a probability measure over \( S_{-i} \), called \( s_i \)'s theory, which induces \( s_i \)'s distribution \( p(\cdot; s_i) \in \Delta(S) \) over the state space by attaching to each \( E \subseteq S \) the probability that \( s_i \)'s theory assigns to \( \{ s_{-i} \in S_{-i} : (s_i, s_{-i}) \in E \} \). The extension from \( s_i \)'s theory to \( p(\cdot; s_i) \) is unique since we assume \( p([s_i]; s_i) = 1 \) and also that \( \text{marg}_{S_{-i}} p(\cdot; s_i) \) coincides with \( s_i \)'s theory (Aumann and Brandenburger, 1995). Intuitively, \( p(\cdot; s_i) \) denotes \( i \)'s conditional beliefs over the state space given the type \( s_i \). A probability measure \( P \in \Delta(S) \) is called a common prior, if for every \( i \in I \) and all \( s_i \in S_i \) the conditional distribution of \( P \) given \( s_i \) coincides with \( p(\cdot; s_i) \).

Belief is formalized in terms of events: the set of states where agent \( i \) believes in \( E \subseteq S \) is defined as

\[
B_i(E) := \{ s \in S : p(E; s_i) = 1 \}.
\]

Then, it is said that \( i \) believes in \( E \) at \( s \), whenever \( s \in B_i(E) \). Note that Aumann and Brandenburger (1995) actually use the term knowledge for probability-1 belief.

An event is mutually believed if everyone believes it. Formally, \( E \subseteq S \) is mutually believed at \( s \), whenever \( s \in B(E) \), where

\[
B(E) := \bigcap_{i \in I} B_i(E).
\]

\(^3\)Our results can be generalized to arbitrary measurable types spaces, similarly to Aumann and Brandenburger (1995, Section 6).
Iterating the mutual belief operator then yields higher-order mutual belief. Formally, $m$-order mutual belief in $E$ is inductively defined by $B^m(E) := B(B^{m-1}(E))$ for all $m > 0$, with $B^1(E) := B(E)$. Then, an event $E$ is commonly believed whenever everyone believes in $E$, everyone believes that everyone believes in $E$, etc. Formally, common belief in $E$ is expressed by the event

$$CB(E) := \bigcap_{m>0} B^m(E).$$

For every player $i \in I$ an action function $a_i : S \rightarrow A_i$ specifies his action at each state, and it is assumed to be $S_i$-measurable, i.e., $a_i(s) = a_i(s')$ if $\{s, s'\} \subseteq [s_i]$, implying that $i$ attaches probability 1 to his actual strategy. The event $[a_i] := \{s \in S : a_i(s) = a_i\}$ contains the states at which agent $i$ plays $a_i$, and $[a_{-i}] := [a_1] \cap \cdots \cap [a_{i-1}] \cap [a_{i+1}] \cap \cdots \cap [a_n]$.

The function $\phi_i : S \rightarrow \Delta(A_{-i})$ specifies $i$’s conjecture at every state, and is defined by

$$\phi_i(s)([a_{-i}]) := p([a_{-i}]; s_i)$$

for each $a_{-i} \in A_{-i}$, and is assumed to be $S_i$-measurable, i.e., $\phi_i(s) = \phi_i(s')$ if $\{s, s'\} \subseteq [s_i]$, implying that $i$ assign probability 1 to his actual conjecture. We define the events $[\phi_i] := \{s \in S : \phi_i(s) = \phi_i\}$ and $[\phi_1, \ldots, \phi_n] := [\phi_1] \cap \cdots \cap [\phi_n]$.

Finally, $g_i : S \rightarrow \mathbb{R}^{|A_i|}$ specifies $i$’s payoff function at each state of the world. Throughout the paper, we assume that $\phi_i$ is also $S_i$-measurable, i.e., $g_i(s) = g_i(s')$ if $\{s, s'\} \subseteq [s_i]$, which implies that $i$ attaches probability 1 to his actual payoff function. For some fixed $g_i : A \rightarrow \mathbb{R}$, let $[g_i] := \{s \in S : g_i(s) = g_i\}$ denote the states where $i$’s payoff function is $g_i$. Then, we also define $[(g_1, \ldots, g_n)] := [g_1] \cap \cdots \cap [g_n]$. We say that there is complete information if $[(g_1, \ldots, g_n)] = S$ for some $(g_1, \ldots, g_n)$.

Furthermore, player $i$ is rational at some state $s$, whenever he maximizes his expected payoff at this state given his conjecture and payoff function. That is,

$$R_i := \left\{ s \in S : a_i(s) \in BR_i(\phi_i(s)) \right\}$$

denotes the event that $i$ is rational. Rationality of all players is then given by the event

$$R := \bigcap_{i \in I} R_i.$$  

2.3. Aumann and Brandenburger’s Epistemic Conditions for Nash equilibrium

In their seminal paper, Aumann and Brandenburger (1995) provided epistemic conditions for Nash equilibrium. Accordingly, if conjectures are derived from a common prior and are commonly believed, while at the same time rationality as well as the payoff functions are mutually believed, then all players
different from \( i \) entertain the same marginal conjecture about \( i \)'s action, and the marginal conjectures constitute a Nash equilibrium of the game. Formally, Aumann and Brandenburger’s epistemic foundation for Nash equilibrium can be stated as follows.

**Theorem 1 (Aumann and Brandenburger (1995))** Let \((I, (A_i)_{i \in I}, (g_i)_{i \in I})\) be a game and \((\phi_1, \ldots, \phi_n)\) be a tuple of conjectures. Suppose that there is a common prior that attaches positive probability to a state \( s \in S \) such that \( s \in B([\langle g_1, \ldots, g_n \rangle]) \cap B(R) \cap CB([\langle \phi_1, \ldots, \phi_n \rangle]) \). Then, there exists a mixed strategy profile \((\sigma_1, \ldots, \sigma_n)\) such that

(i) \( \text{marg}_{A_i} \phi_j = \sigma_i \) for all \( j \in I \setminus \{i\} \),

(ii) \((\sigma_1, \ldots, \sigma_n)\) is a Nash equilibrium of \((I, (A_i)_{i \in I}, (g_i)_{i \in I})\).

Subsequently, Polak (1999) showed that in complete information games, common belief in conjectures and mutual belief in rationality entail common belief in rationality. In the context of Theorem 1, Polak’s result implies that without common belief in rationality being present, sufficient conditions for Nash equilibrium must fail to satisfy common belief in conjectures or mutual belief in rationality. In fact, we will weaken both assumptions in Section 4 below and yet obtain Nash equilibrium.


3. **PAIRWISE INTERACTIVE BELIEF**

The standard intuitive explanation for the emergence of common belief is based on public announcement. Accordingly, once an event is publicly announced it becomes common belief in the sense that not only everyone believes in it, but also everyone believes that everyone in believes it, etc. Note that for mutual belief to obtain, the agents are only required to each believe in the event, and hence mere private announcements suffice.

Yet, an event may be publicly (privately) announced to some but not all players. For instance, an event could be publicly (privately) announced to \( Alice \) and \( Bob \), but not to \( Claire \). Common belief (mutual belief) in the event between \( Alice \) and \( Bob \) would then emerge, but not necessarily common belief (mutual belief). Due to such epistemic possibilities we now introduce pairwise interactive belief operators.

Let \( E \subseteq S \) be some event and \( i, j \in I \) be two players. We say that \( E \) is **pairwise mutually believed** between \( i \) and \( j \) whenever they both believe in \( E \). Formally, pairwise mutual belief in \( E \) between \( i \) and
j is denoted by the event

\[ B_{i,j}(E) := B_i(E) \cap B_j(E). \]

Note that mutual belief implies pairwise mutual belief, but not conversely. We say that \( E \) is *pairwise commonly believed* between \( i \) and \( j \) whenever \( E \) is commonly believed between them. Formally, inductively let \( B_{i,j}^m(E) := B_{i,j}(B_{i,j}^{m-1}(E)) \) for each \( m > 0 \) with \( B_{i,j}^1(E) := B_{i,j}(E) \). Pairwise common belief in \( E \) between \( i \) and \( j \) is then defined as the event

\[ CB_{i,j}(E) := \bigcap_{m>0} B_{i,j}^m(E). \]

Observe that common belief implies pairwise common belief, but not conversely.

In contrast to the standard notions of mutual and common belief, our two pairwise epistemic operators describe interactive belief only locally for pairs of agents, postulating the existence of exclusively binary relations of epistemic relevance. Formally, we represent a set of such binary relations by means of an undirected graph \( G = (I, E) \), where the set of vertices \( I \) denotes the set of players, and the set of edges \( E \) describe binary symmetric relations \((i, j) \in I \times I\) between pairs of players.

In principle, the graph \( G \) does neither enrich the epistemic model nor add any additional structure to the game whatsoever, but only provides a formal framework for expressing pairwise local conditions of interactive belief, e.g. a graph containing an edge between \( i \) and \( j \) but not between \( j \) and \( k \) can be used to model a situation where an event is pairwise mutually believed between \( i \) and \( j \) but not between \( j \) and \( k \). Thus, the connectedness of two agents by an edge is of purely epistemic and not physical character. However, \( G \) could also be interpreted as a network.\(^4\)

Next, some graph theoretic notions are recalled. A sequence \((i_k)_{k=1}^m\) of players is a *path* whenever \((i_k, i_{k+1}) \in E\) for all \( k \in \{1, \ldots, m - 1\} \), i.e. in a path every two consecutive players are linked by an edge. Moreover, a graph \( G \) is called *connected* if it contains a path \((i_k)_{k=1}^m\) such that for every \( i \in I \) there is some \( k \in \{1, \ldots, m\} \) with \( i_k = i \). Besides, \( G \) is *Hamiltonian*, whenever there exists a path \((i_k)_{k=1}^n\) such that for every \( i \in I \) there is a unique \( k \in \{1, \ldots, n\} \) with \( i_k = i \), and also \((i_1, i_n) \in E\). Intuitively, a Hamiltonian graph contains a cycle in which each player appears exactly once. In addition, \( G \) is *complete*, if \((i, j) \in E\) for all \( i, j \in I\).

Two specific pairwise-local epistemic conditions are now introduced.

**Definition 1** Let \((I, (A_i)_{i \in I}, (g_i)_{i \in I})\) be a game, \( G \) be an undirected graph, \( s \) be a state, and \((\phi_1, \ldots, \phi_n)\) be a tuple of conjectures.

\(^4\)See Section 5.
• **Rationality is G-pairwise mutually believed** at \( s \), whenever \( s \in B_{i,j}(R_i \cap R_j) \) for all \((i, j) \in E\).

• **Payoffs are G-pairwise mutually believed** at \( s \), whenever \( s \in B_{i,j}([g_i] \cap [g_j]) \) for all \((i, j) \in E\).

• **Conjectures are G-pairwise commonly believed** at \( s \), whenever \( s \in CB_{i,j}([\phi_i] \cap [\phi_j]) \) for all \((i, j) \in E\).

Note that henceforth an edge between two agents \( i \) and \( j \) in a graph \( G \) signifies that \( i \) and \( j \) entertain both pairwise mutual belief in rationality and payoffs as well as pairwise common belief in conjectures.

The standard notions of mutual belief in rationality, mutual belief in payoffs and common belief in conjectures, which are also used by Aumann and Brandenburger (1995), are weakened by G-pairwise mutual belief in rationality, G-pairwise mutual belief in payoffs and G-pairwise common belief in conjectures, respectively. Formally, observe that

\[
B(R) = \bigcap_{i \in I} B_i(R_1 \cap \cdots \cap R_n)
\]

\[
\subseteq \bigcap_{i \in I} \bigcap_{j \in I: (i, j) \in E} B_{i,j}(R_1 \cap \cdots \cap R_n)
\]

\[
\subseteq \bigcap_{i \in I} \bigcap_{j \in I: (i, j) \in E} B_{i,j}(R_i \cap R_j),
\]

as well as

\[
B([g_1, \ldots, g_n]) \subseteq \bigcap_{i \in I} \bigcap_{j \in I: (i, j) \in E} B_{i,j}([g_i] \cap [g_j]).
\]

and

\[
CB([\phi_1, \ldots, \phi_n]) \subseteq \bigcap_{i \in I} \bigcap_{j \in I: (i, j) \in E} CB_{i,j}([\phi_i] \cap [\phi_j]).
\]

Indeed, our concepts are considerably weaker than the standard notions on two distinct dimensions. Firstly, the events rationality, payoffs and conjectures in Definition 1 only refer to the rationality, the payoffs and the conjectures of the two connected agents. Secondly, our two pairwise-local epistemic conditions impose epistemic restrictions only on the pairs of connected players in the graph, whereas standard interactive belief does so across all pairs of players. In fact, mutual belief and common belief coincide with G-pairwise mutual belief and G-pairwise common belief, respectively, whenever \( G \) is complete.

The following example illustrates the two new concepts of G-pairwise mutual belief in rationality and G-pairwise common belief in conjectures in a game with complete information, and also relates them to the standard notions of mutual belief in rationality and common belief in conjectures.
Example 1 Consider the coordination game \((I, (A_i)_{i \in I}, (g_i)_{i \in I})\), where \(I = \{Alice, Bob, Claire, Donald\}\), \(A_i = \{h, \ell\}\) for all \(i \in I\), and
\[
g_i(a_A, a_B, a_C, a_D) = \begin{cases} 
2 & \text{if } a_i = h \text{ for all } i \in I, \\
1 & \text{if } a_i = \ell \text{ for all } i \in I, \\
0 & \text{otherwise.}
\end{cases}
\]

Now, assume the type spaces
\[
S_A = \{s^1_A(\ell), s^2_A(\ell)\}, \\
S_B = \{s^1_B(\ell), s^2_B(\ell), s^3_B(h)\}, \\
S_C = \{s^1_C(\ell), s^2_C(h)\}, \\
S_D = \{s^1_D(h), s^2_D(\ell), s^3_D(h)\},
\]
with the action in parenthesis denoting the respective player \(i\)’s action at every state given by the function \(a_i\). Moreover, suppose that the players have a common prior
\[
P \text{ uniformly distributed over } \{ (s^1_A, s^1_B, s^1_C, s^1_D), (s^2_A, s^2_B, s^1_C, s^2_D), (s^2_A, s^3_B, s^2_C, s^3_D) \}.
\]

For instance, if Alice’s type is \(s^2_A\), then she attaches probability \(\frac{1}{2}\) to \((s^2_A, s^2_B, s^1_C, s^2_D)\) and \(\frac{1}{2}\) to \((s^2_A, s^3_B, s^2_C, s^3_D)\).

Let \(G = (I, \mathcal{E})\) be a Hamiltonian graph such that
\[
I = \{Alice, Bob, Claire, Donald\}, \\
\mathcal{E} = \{(Alice, Bob), (Bob, Claire), (Claire, Donald), (Donald, Alice)\}.
\]

Consider the state \((s^1_A, s^1_B, s^1_C, s^1_D)\) which receives positive probability by the common prior, and observe that the conjectures are \(G\)-pairwise commonly believed. Indeed, Alice’s and Bob’s conjectures – Alice being certain that each of her opponents plays \(\ell\) and Bob being certain that each of his opponents plays \(\ell\) – are pairwise commonly believed between them. Also, conjectures are pairwise commonly believed between Bob and Claire, between Claire and Donald, as well as between Donald and Alice. However, conjectures are not commonly believed. In fact, they are not even mutually believed, as Claire does not believe in Alice’s conjecture at \((s^1_A, s^1_B, s^1_C, s^1_D)\): given her type \(s^1_C\), she attaches probability of only \(\frac{1}{2}\) to Alice being certain that each of her opponents chooses \(\ell\).

Furthermore, note that rationality is \(G\)-pairwise mutually believed at \((s^1_A, s^1_B, s^1_C, s^1_D)\). However, it is not mutually believed at \((s^1_A, s^1_B, s^1_C, s^1_D)\) that everyone is rational. Indeed, Claire does not believe that Alice is rational at \((s^1_A, s^1_B, s^1_C, s^1_D)\), since \(\ell\) is not a best response for Alice at the state \((s^1_A, s^1_B, s^1_C, s^1_D)\) which receives positive probability by \(s^1_A\)’s distribution.

Besides, observe that for every \(i \in I\), the remaining players share the same marginal conjecture about \(i\)’s action at \((s^1_A, s^1_B, s^1_C, s^1_D)\), i.e. \(\text{margin}_A(\phi_j(s^1_A, s^1_B, s^1_C, s^1_D)) = \sigma_i\) for all \(j \in I \setminus \{i\}\), where the mixed
strategy $\sigma_i$ assigns probability 1 to $i$ playing $\ell$. Also, $(\ell, \ell, \ell, \ell)$ constitutes a Nash equilibrium of the game.

In the preceding example neither the conjectures nor rationality are mutually believed. Hence, the two central elements of Aumann and Brandenburger’s conditions for Nash equilibrium are violated, yet both of their conclusions do hold. In fact, players entertain the same marginal conjectures about their opponents’ strategies, and also these marginal conjectures form a Nash equilibrium. On the basis of Example 1 the natural question then arises, whether there exists a general relation between $G$-pairwise mutual belief in rationality and $G$-pairwise common belief in conjectures on the one hand, and Nash equilibrium on the other hand.

4. PAIRWISE INTERACTIVE KNOWLEDGE AND NASH EQUILIBRIUM

We now weaken Aumann and Brandenburger’s conditions for Nash equilibrium by means of pairwise interactive belief. Indeed, the following result shows that $G$-pairwise mutual belief rationality and payoffs and $G$-pairwise common belief in conjectures already suffice for Nash equilibrium, if $G$ is Hamiltonian.

**Theorem 2** Let $(I, (A_i)_{i \in I}, (g_i)_{i \in I})$ be a game, $G$ be a Hamiltonian undirected graph and $(\phi_1, \ldots, \phi_n)$ be a tuple of conjectures. Suppose that there is a common prior that attaches positive probability to a state $s \in S$ such that $s \in B_{i,j}([g_i] \cap [g_j]) \cap B_{i,j}(R_i \cap R_j) \cap CB_{i,j}([\phi_i] \cap [\phi_j])$ for all $(i, j) \in \mathcal{E}$. Then, there exists a mixed strategy profile $(\sigma_1, \ldots, \sigma_n)$ such that

(i) $\text{marg}_{A_i} \phi_j = \sigma_i$ for all $j \in I \setminus \{i\}$,

(ii) $(\sigma_1, \ldots, \sigma_n)$ is a Nash equilibrium of $(I, (A_i)_{i \in I}, (g_i)_{i \in I})$.

**Proof:** Let $G = (I, \mathcal{E})$ be a Hamiltonian graph, and suppose for sake of simplicity and without loss of generality that $(i, i + 1) \in \mathcal{E}$ for all $i \in \{1, \ldots, n - 1\}$, and also $(n, 1) \in \mathcal{E}$.

Proof of (i). Let $(i, j) \in \mathcal{E}$ and define $F_{i,j} := CB_{i,j}([\phi_i] \cap [\phi_j])$. Firstly, observe that by Aumann and Brandenburger (1995, Lem. 4.4),

\[ (1) \quad F_{i,j} \subseteq B_i(F_{i,j}). \]

Secondly, note that

\[ F_{i,j} \subseteq B_i([\phi_i] \cap [\phi_j]) \]

\[ \subseteq B_i([\phi_i]) \quad \text{(by Aumann and Brandenburger (1995, Lem. 4.3))} \]

\[ = [\phi_i] \quad \text{(by Aumann and Brandenburger (1995, Lem. 2.6))} \]

\[ (2) \]

\[ \subseteq B_i([\phi_i]) \quad \text{(by Aumann and Brandenburger (1995, Lem. 4.3))} \]

\[ = [\phi_i] \quad \text{(by Aumann and Brandenburger (1995, Lem. 2.6))} \]
which implies that for every $a_{-i} \in A_{-i}$

(3) \[ p([a_{-i}]; s'_i) = \phi_i(a_{-i}) \text{ for all } s' \in F_{i,j}. \]

Therefore, if we combine (1) and (3) with the fact that $P$ is a common prior, it follows from Aumann and Brandenburger (1995, Lem. 4.5) that for an arbitrary $a_k \in A_k$ of some $k \in I \setminus \{i, j\}$,

\[ p([a_k]; s_i) = \frac{P([a_k] \cap F_{i,j})}{P(F_{i,j})}, \]

which is a well-defined conditional probability as $s \in F_{i,j}$, and therefore $P(F_{i,j}) \geq P(s) > 0$. Similarly, the previous equation holds for $j$, i.e.,

\[ p([a_k]; s_j) = \frac{P([a_k] \cap F_{i,j})}{P(F_{i,j})}. \]

thus obtaining $p([a_k]; s_i) = p([a_k]; s_j)$ for all $a_k \in A_k$, which implies $\text{marg}_{A_k} \phi_i = \text{marg}_{A_k} \phi_j$. Finally, recall that $G$ is such that $\{(1, 2), \ldots, (n-1, n), (n, 1)\} \subseteq \mathcal{E}$. Then, it follows from repeatedly applying the previous step that

\[
\text{marg}_{A_k} \phi_{k+1} = \text{marg}_{A_k} \phi_{k+2} = \cdots = \text{marg}_{A_k} \phi_n \\
= \text{marg}_{A_k} \phi_1 = \cdots = \text{marg}_{A_k} \phi_{k-1} \\
=: \sigma_k.
\]

**Proof of (ii).** Firstly, we show that for every $E \subseteq S$,

(4) \[ p([a_i] \cap E; s_i) = p([a_i]; s_i) \cdot p(E; s_i). \]

Recall that for every $a_i \in A_i$, either $[s_i] \subseteq [a_i]$ and thus $p([a_i]; s_i) = 1$, or $[s_i] \cap [a_i] = \emptyset$ and hence $p([a_i]; s_i) = 0$. If $[s_i] \subseteq [a_i]$, it follows that

\[ p([a_i] \cap E; s_i) = p\left(\{s_{-i} \in S_{-i} : (s_i, s_{-i}) \in [a_i] \cap E\}; s_i\right) \\
= p\left(\{s_{-i} \in S_{-i} : (s_i, s_{-i}) \in [s_i] \cap E\}; s_i\right) \\
= p(E; s_i) \\
= p([a_i]; s_i) \cdot p(E; s_i). \]

If on the other hand, $[s_i] \cap [a_i] = \emptyset$, it follows that

\[ p([a_i] \cap E; s_i) = p\left(\{s_{-i} \in S_{-i} : (s_i, s_{-i}) \in [a_i] \cap E\}; s_i\right) = 0 \\
= p([a_i]; s_i) \cdot p(E; s_i), \]

and therefore (4) holds.
Next, we show that for all $i \in I$,
\begin{equation}
\phi_i = \sigma_1 \times \cdots \sigma_{i-1} \times \sigma_{i+1} \times \cdots \times \sigma_n.
\end{equation}
Without loss of generality, we prove (5) for player 1. Consider an arbitrary $a_{-1} = (a_2, \ldots, a_n)$, and observe that
\[
\phi_1(a_{-1}) = P([a_2] \cap \cdots \cap [a_n]; s'_1) = \sum_{s' \in F_{1,2}} p([a_2] \cap \cdots \cap [a_n]; s'_1) \cdot P(s'|F_{1,2})
\]
\[
= P([a_2] \cap \cdots \cap [a_n] \mid F_{1,2}) = \sum_{s' \in F_{1,2}} p([a_2] \cap \cdots \cap [a_n]; s'_2) \cdot P(s'|F_{1,2})
\]
\[
= p([a_3] \cap \cdots \cap [a_n]; s_2) \sum_{s' \in F_{1,2}} p([a_2]; s'_2) \cdot P(s'|F_{1,2})
\]
\[
= p([a_3] \cap \cdots \cap [a_n]; s_2) \cdot P([a_2] \mid F_{1,2}) = \phi_2(a_3, \ldots, a_n) \cdot P([a_2] \mid F_{1,2})
\]
\[
= \phi_2(a_3, \ldots, a_n) \sum_{s' \in F_{1,2}} p([a_2]; s'_1) \cdot P(s'|F_{1,2})
\]
\[
= \phi_2(a_3, \ldots, a_n) \cdot P([a_2]; s_1) \sum_{s' \in F_{1,2}} P(s'|F_{1,2})
\]
\[
= \phi_2(a_3, \ldots, a_n) \cdot \phi_1(a_2)
\]
Repeating the previous step inductively yields
\[
\phi_1(a_2, \ldots, a_n) = \phi_1(a_2) \cdots \phi_{n-1}(a_n).
\]
Now, recall from (i) that all players agree on the marginal conjectures, implying that
\[
\phi_1(a_2, \ldots, a_n) = \phi_1(a_2) \cdots \phi_1(a_n),
\]
and therefore (5) obtains. Finally, $a_i \in BR_i(\phi_i)$ for all $a_i \in supp(\phi_i)$ follows directly from applying Aumann and Brandenburger (1995, Lem. 4.2) to all pairs of connected players. Q.E.D.

The contribution of the previous result to the epistemic foundation of Nash equilibrium is twofold.

Firstly, we significantly relax the standard epistemic conditions of Aumann and Brandenburger (1995), by no longer requiring neither common belief in conjectures nor mutual belief in rationality nor mutual belief in the utilities.
Secondly, Theorem 2 offers further insight on the relation between Nash equilibrium and common belief in rationality. In fact, for many years the predominant view suggested that common belief in rationality was an essential element of Nash equilibrium. This view was then challenged by Aumann and Brandenburger (1995) who required only mutual belief in rationality in their foundation for Nash equilibrium. However, Polak (1999) observed more recently that Aumann and Brandenburger’s conditions actually do imply common belief in rationality. In a sense, his result thus restored some of the initial confidence in the importance of common belief in rationality in the context of Nash equilibrium. Our theorem not only confirms Aumann and Brandenburger’s initial intuition about the non-necessity of common belief in rationality for Nash equilibrium, but also provides sufficient conditions for Nash equilibrium that do not even imply mutual belief in rationality. To see this, consider Example 1, and observe that at \((s_1^A, s_2^B, s_1^C, s_1^D)\), which satisfies all the conditions of our theorem, Claire does not believe that Alice is rational, as \((s_1^A, s_2^B, s_1^C, s_1^D) \notin B_C(R_A)\).

5. DISCUSSION

Tightness. The assumption of the graph being Hamiltonian is crucial for Theorem 2. Indeed, it is now shown by means of an example that the graph simply being connected does not suffice for the conclusions of Theorem 2 to obtain, even if payoffs and rationality are commonly believed. In that sense our epistemic foundation is tight.

EXAMPLE 2 Consider the anti-coordination game \((I, (A_i)_{i \in I}, (g_i)_{i \in I})\), where \(I = \{Alice, Bob, Claire\}\), \(A_i = \{h, \ell\}\) for all \(i \in I\), and

\[
g_i(a_A, a_B, a_C) = \begin{cases} 0 & \text{if } a_A = a_B = a_C, \\ 1 & \text{otherwise.} \end{cases}
\]

Now, assume the type spaces

\[
S_A = \{s_1^A(h), s_2^A(h)\},
\]

\[
S_B = \{s_1^B(h), s_2^B(\ell)\},
\]

\[
S_C = \{s_1^C(\ell)\},
\]

with the action in parenthesis denoting the respective player \(i\)’s action at every state given by the function \(a_i\). Moreover, assume that the players have a common prior

\[
P \text{ uniformly distributed over } \{ (s_1^A, s_1^B, s_1^C), (s_2^A, s_2^B, s_1^C) \}.
\]

Let \(G = (I, \mathcal{E})\) be a connected graph such that

\[
I = \{Alice, Bob, Claire\},
\]

\[
\mathcal{E} = \{(Alice, Bob), (Bob, Claire)\}.
\]
Note that at every $s \in S$, rationality is commonly believed, and conjectures are $G$-pairwise commonly believed. Moreover, at $(s_A^1, s_B^1, s_C^1)$, Alice is certain that Bob chooses $h$ and Claire chooses $\ell$, whereas Claire’s conjecture attaches probability $\frac{1}{2}$ to both of her opponents playing $h$, and $\frac{1}{2}$ to Alice playing $h$ and Bob playing $\ell$. Therefore, Alice and Claire disagree on their marginal conjecture about Bob’s action, implying that the conclusion of Theorem 2 does not hold. In fact, all conditions of Theorem 2 are satisfied apart from $G$ being Hamiltonian. Hence, $G$ simply being connected instead of Hamiltonian does not suffice for Nash equilibrium.

In general, the conclusions of Theorem 2 fail for a merely connected – but not Hamiltonian – graph, because in order for players $i$ and $j$ to agree on their marginal conjectures about a third player $k$, there must exist a path connecting $i$ and $j$ which does not pass through $k$. Otherwise, it cannot be inductively established that $i$ and $j$ have the same marginal conjecture about $k$’s strategies.

Emergence of pairwise epistemic conditions. As already mentioned, in principle the graph $G$ does not add any additional structure to the game and is only used to describe pairwise epistemic conditions. However, $G$ can also be interpreted as a network. In this case, our conditions can be perceived as a steady state of a sequence of private communication correspondences between connected agents, e.g. similarly to Parikh and Krasucki (1990). This is particularly interesting for games with many players, such as large economies where agents may learn relevant personal characteristics – such as rationality, conjectures or preferences – of their neighbors only. However, note that our aim is not to explicitly model how pairwise interactive belief emerges in a dynamic setting, but rather to study the players’ beliefs and actions, once convergence to such a state has already occurred.

Knowledge and belief. Following Aumann and Brandenburger (1995), we provide epistemic conditions for Nash equilibrium in terms of probability-1 belief, instead of knowledge. Hence, players are not required to satisfy the truth axiom, implying that players may hold false beliefs. In particular, player $i$ may wrongly attach probability 1 to the event that $j$’s conjecture is $\phi_j$. In an earlier version of this paper, we prove Theorem 2 in a partitional model using knowledge instead of probability-1 belief (Bach and Tsakas, 2012). Note that such an approach does not restrict the state space to have a product structure as opposed to the type-based one employed here.

Belief in an opponent’s conjecture. Already Aumann and Brandenburger (1995) recognize the conceptual difficulty in assuming belief in an opponent’s conjecture. We do not intend to provide any remedy to this
problematic assumption whatsoever, as we also assume that players believe in the conjectures of some of their opponents. Thus, the conceptual issue imposed by assuming belief in opponents’ conjectures persists. However, we show that less belief about the opponents’ conjectures is actually needed for Nash equilibrium to obtain.

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