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The reflection effect for constant risk averse agents

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Abstract

Assume a decision maker has a preference relation over monetary lotteries. The reflection effect, first observed by Kahneman and Tversky, states that the preference order for two lotteries is reversed once they are multiplied by -1. The decision maker is constant risk averse (CRA) if adding the same constant to two distributions, or multiplying them by the same positive constant, will not change the preference relation between them. We combine these two axioms with the betweenness axiom and continuity, and prove a representation theorem. A technical curiosity is that the functions we get satisfy the betweenness axiom, yet are not necessarily Gâteaux (nor Fréchet) differentiable.

Keywords: Reflection effect; Betweenness; Constant risk aversion

1. Introduction

We consider a decision maker who has a preference relation over the set of monetary lotteries. In their seminal paper, Kahneman and Tversky (1979) provide evidence that decision makers who prefer being paid some lottery *A* over another lottery *B* will reverse their preference when paying, i.e., will prefer to pay lottery *B* over *A*. Kahneman and Tversky name their observation 'the reflection effect'.[1]

In this paper we consider decision makers who adhere with the reflection effect as well as some other well known axioms, such as being constant risk averse and the betweenness axiom. Constant risk aversion (CRA) implies that adding the same constant to two lotteries, or multiplying them by the same positive constant, will not change the

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[1] An alternative way to consider the reflection effect is in terms of certainty equivalents of a distribution. The reflection effect implies that the certainty equivalent for some distribution of prizes *X* is minus that of the distribution $-X$.

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preference relation between them. The betweenness axiom, introduced by Coombs and Huang (1976), states that if a lottery, *A*, is preferred over another lottery, *B*, then a lottery over these two lotteries is also preferred over *B*. The main result of this paper is in providing a representation theorem for all three axioms (reflection, CRA and betweenness), combined with a continuity property.

It turns out that the family of functionals we get in the representation theorem is quite small, and can be parameterized by one variable. The expected value is, obviously, one of the family members. Comparing the results obtained here with those of Dekel (1986) and Chew (1989), who focus on the betweenness property as an alternative to Savage's (1954) controversial independence axiom, leads to the conclusion that the reflection effect and CRA reduces the family of functionals quite drastically. We note that the functionals we obtain are a special case of their functionals.

Functionals that comply with CRA have been well known in the expected as well as nonexpected utility framework. Actually, within the expected utility framework CRA implies expected value maximization. Examples of functionals which satisfy CRA within the nonexpected utility framework are Yaari's (1987) dual theory and Roberts (1980). Safra and Segal (1998) provide a set of representation theorems when CRA is combined with various other well known axioms such as mixture symmetry, quasi-concavity, quasi-convexity, zero independence and others. A recent unpublished manuscript by Wakker and Zank (1997) provides an axiomatization of Prospect theory, and thus in particular the reflection effect. Similar to our paper constant proportional risk aversion is assumed. However, as other aspects of cumulative Prospect theory are assumed on the one hand and betweenness is not assumed on the other hand, their results differ from ours.

A technical curiosity we get as a corollary is pointed out in Safra and Segal (1998). The functionals we get are an example of functionals which satisfy the betweenness axiom yet are not Gâteaux (nor Fréchet) differentiable. This shows that betweenness is not enough to ensure differentiability, which is a property assumed often in the nonexpected utility literature (e.g. Machina, 1982).

The paper is organized as follows. In Section 2 the model and main result are introduced. The proof of the main result is given in Section 3, and Section 4 provides some concluding remarks.

2. The model and main result

Let $\mathscr{D}$ be the set of all bounded random variables on $\mathbb{R}$, representing lotteries of gains (or losses). For any $X \in \mathscr{D}$, let $F_X(\cdot)$ denote its cumulative distribution function (cdf). For $c, d \in \mathbb{R}$, $c > 0$, let $Y = cX + d$ be the element of $\mathscr{D}$ satisfying $F_Y(t) = F_X((t-d)/c)$. Also, let $-X$ denote the element of $\mathscr{D}$ satisfying $F_{-X}(t) = 1 - F_X(-t)$. For any $X, Y \in \mathbb{R}$ and $\alpha \in [0, 1]$ let $Z = \alpha X \oplus (1-\alpha)Y$ be the random variable satisfying $F_Z(t) = \alpha F_X(t) + (1-\alpha)F_Y(t)$.

Let $\succeq$ be a complete transitive preference relation on $\mathscr{D}$ and let $\sim$ be the indifference relation it induces. Let $f:\mathscr{D} \to \mathbb{R}$, defined implicitly by the equation $X \sim \delta_{f(X)}$ (where δ_a

denotes the random variable 'a for sure') be the certainty equivalent functional induced by $\succeq$.

The purpose of this work is to map such certainty equivalent functionals which satisfy certain properties:

- The Oddness Axiom (O): f is an odd function, i.e., for all $X \in \mathcal{D}$, $f(-X) = -f(X)$. This axiom is motivated by the reflection effect.[2]
- Constant Risk Aversion (CRA): for all $X \in \mathcal{D}$ $c > 0$ and $d \in \mathbb{R}$, $f(cX + d) = cf(X) + d$.[3]
- Betweenness (B): For all $X, Y \in \mathcal{D}$, such that $f(X) < f(Y)$, and for any $\alpha \in (0,1)$, $f(X) < f(\alpha X \oplus (1-\alpha)Y) < f(Y)$.
- Continuity (C): f is continuous w.r.t. the weak topology on $\mathcal{D}$.

Note that combining the axioms (CRA) and (O) gives full invariance w.r.t. affine transformations. Namely, for all $X \in \mathcal{D}$ c, $d \in \mathbb{R}$, $f(cX + d) = cf(X) + d$.

We refer to functions $f:\mathcal{D} \to \mathbb{R}$, satisfying (C), (CRA), (O) and (B), as Fair Solutions (FS). In order to clearly present our results we define the following family of real valued functions on $\mathcal{D}$. Let $\phi_c(X) = \text{argmin}_t(E|X - t|^{c+1})$.

Theorem A. *$f:\mathcal{D} \to \mathbb{R}$ is a Fair Solution if and only if there exists $c \in \mathbb{R}_+$ such that $f = \phi_c$.*

Remark 1. *For $c = 1$, ϕ_c is the expectation operator.*

Remark 2. *Taking $c \to 0$ implies that ϕ_c converges to the median. Note that the median itself does not satisfy continuity.*

3. Proof of main result

Throughout the proofs we shall be using the following properties of fair solutions:

- Indifference (I): For any X, $Y \in \mathcal{D}$, $f(X) = f(Y)$ implies $f(X) = f(\alpha X \oplus (1-\alpha)Y) = f(Y)$, for any $\alpha \in (0,1)$.
- Consistency (Con): If $X \in \mathcal{D}$ satisfies $\text{prob}(X = a) = 1$ for some $a \in \mathbb{R}$, then $f(X) = a$.

Lemma 1. *Axioms (C), (B), (CRA) and (O) imply (I) and (Con).*

Proof. We prove the claim in two parts. In part (i) we show that (C), (B), (CRA) and (O) imply (I), and in part (ii) we claim that (Con) follows from (I).

Part (i): Suppose $f(X) = f(Y)$ and that $f(X) > f(\alpha X \oplus (1-\alpha)Y)$ for some $\alpha \in (0, 1)$. Let $Z_n = X + (1/n)$. And let $W_n = \alpha Z_n \oplus (1-\alpha)Y$. By (CRA) $f(Z_n) > f(X)$, so by

[2] In terms of the preference relation: $X \succeq Y \Rightarrow -X \preceq -Y$.

[3] In terms of the preference relation: $X \succeq Y \Rightarrow aX + b \succeq aY + b$ for a, $b \in \mathbb{R}$, $a > 0$.

betweenness $f(W_n) > f(Y)$. Applying (C) yields $f(W_n) \to f(\alpha X \oplus (1-\alpha)Y)$ and so $f(W_n) \to f(\alpha X \oplus (1-\alpha)Y) \geq f(Y) = f(X)$ which is a contradiction.

Part (ii): For any $a \in \mathbb{R}$ denote by X_a the random variable satisfying prob$(X = a) = 1$. *By* (*CRA*) $f(X_0) = f(2X_0) = 2f(X_0)$. *Therefore*, $f(X_0) = 0$. *Again by* (*CRA*) $f(X_a) = f(X_0) + a = a$. □

We begin the proof of our main result by focusing on atomic random variables. We shall use the notation

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$$

to denote the random variable which has atoms $\{a_i\}_{i=1}^n$ and assigns measure $p_i/\Sigma_i^n\, p_j$ to a_i, where $a_i \in \mathbb{R}$ and $p_i \in \mathbb{R}_+$. The following definition is extensively used throughout the proof:

Definition. For any function ϕ: $\mathscr{D} \to \mathbb{R}$, define the *underlying function* f_ϕ:$[0,1] \to \mathbb{R}$ as follows:

$$f_\phi\,(p) = \phi\begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}.$$

The claim in Theorem A has two directions. The first is that any function of the form ϕ_c satisfies (C), (CRA), (O) and (B). To show this, we must first show that ϕ_c is well defined (see Lemma 2) and then proceed to ensure that it satisfies all axioms (Lemma 3). The second direction is more involved. In this direction, we make use of the underlying functions just defined. The proof of the second direction proceeds in steps as follows:

Step I. We show that the underlying function of ϕ_c is $f_{\phi_c}(p) = p^{1/c}/(p^{1/c} + (1-p)^{1/c})$ (Lemma 4).

For convenience, we use the notation

$$f^c(p) := \left(\frac{p^{1/c}}{p^{1/c} + (1-p)^{1/c}}\right)$$

throughout.

Step II. We show that any function not of this form cannot be an underlying function for a fair solution (Lemma 7). Finally,

Step III. We show that any two different fair solutions must have different underlying functions (Lemma 8). Thus, we conclude that there is a one-to-one correspondence between fair solutions and underlying functions.

We now turn to the lemmas:

Lemma 2. *For any* $c>0$ *and any* $X\in\mathscr{D}$, $\phi_c(X)$ *is uniquely determined.*

Proof. For any given $X\in\mathscr{D}$, let $g_X(t)=\int_{w>t}(w-t)^c\,\mathrm{d}F_X(w)$ and $h_X(t)=\int_{w\le t}(t-w)^c\,\mathrm{d}F_X(w)$. For $c>0$, it is quite obvious that $g_X(t)$ is strictly decreasing as a function of t, while $h_X(t)$ is strictly increasing. As these functions are both positive and as $\lim_{t\to-\infty}h_X(t)=\lim_{t\to\infty}g_X(t)=0$, we conclude that there is exactly one value, t_0, where they are equal. By differentiating $E|X-t|^{c+1}$ by t for $c>0$ we can see that the minimum is obtained exactly at that same value, t_0. We conclude that ϕ_c is uniquely determined. □

We turn to show that ϕ_c is indeed a fair solution:

Lemma 3. *For any* $c\in\mathbb{R}$, ϕ_c *satisfies* (*C*), (*CRA*), (*O*) *and* (*B*).

Proof. (C), (CRA) and (O) are straightforward, so it remains to show that ϕ_c satisfies (B).

Suppose not, i.e., there exist $X, Y\in\mathscr{D}$, $\phi_c(X)<\phi_c(Y)$, and $\alpha\in(0, 1)$ such that

$$\phi_c(\alpha X\oplus(1-\alpha)Y)\ge\phi_c(Y)\ . \tag{1}$$

Using the notation from the proof of Lemma 2, we obtain

$$g_{\alpha X\oplus(1-\alpha)Y}(\phi_c(\alpha X\oplus(1-\alpha)Y)=h_{\alpha X\oplus(1-\alpha)Y}(\phi_c(\alpha X\oplus(1-\alpha)Y))\ . \tag{2}$$

Note that for a fixed t, $g_{\alpha X\oplus(1-\alpha)Y}(t)=\alpha g_X(t)+(1-\alpha)g_Y(t)$, and similarly for $h_\bullet(t)$. Combining this with (1) and the monotonicity of $g_Z(\cdot)$ and $h_Z(\cdot)$ (for any fixed Z), we get

$$\begin{aligned}&g_{\alpha X\oplus(1-\alpha)Y}(\phi_c(\alpha X\oplus(1-\alpha)Y))=\alpha g_X(\phi_c(\alpha X\oplus(1-\alpha)Y))\\&\quad+(1-\alpha)g_Y(\phi_c(\alpha X\oplus(1-\alpha)Y))<\alpha g_X(\phi_c(X))+(1-\alpha)g_Y(\phi_c(Y))\\&=\alpha h_X(\phi_c(X))+(1-\alpha)h_Y(\phi_c(Y))\\&<\alpha h_X(\phi_c(\alpha X\oplus(1-\alpha)Y))+(1-\alpha)h_Y(\phi_c(\alpha X\oplus(1-\alpha)Y))\\&=h_{\alpha X\oplus(1-\alpha)Y}(\phi_c(\alpha X\oplus(1-\alpha)Y)),\end{aligned}$$

which contradicts Eq. (2). □

Lemmas 2 and 3 have provided one direction of the main result, i.e. that all functions of the form ϕ_c are indeed fair solutions. We turn to show the other direction. We do this by focusing on underlying functions:

Lemma 4. *The underlying function of* ϕ_c *is* $f_{\phi_c}=f^c$.

Proof. Follows from definitions, by differentiation. □

Lemma 5. *Assume* ϕ *satisfies* (*C*), (*CRA*), (*O*), *and* (*B*). *Then the following holds for its underlying function*:

1. $f_\phi(0)=0$, $f_\phi(1/2)=(1/2)$, $f_\phi(1)=1$ (*follows from* (*CRA*) *and* (*O*)).
2. $f_\phi[0,\ 1]\subset[0,\ 1]$ (*follows from* (*B*)).
3. f_ϕ *is monotonically increasing* (*follows from* (*B*)).
4. $f_\phi(p)+f_\phi(1-p)=1,\ \forall\ p\in[0,1]$ (*follows from* (*CRA*) *and* (*O*)).
5. $f_\phi(\cdot)$ *is continuous* (*follows from* (*C*)).

Lemma 6. If f:[0,1]→[0,1] has the properties listed in Lemma 5, then either $\exists\, c\in(0,\infty)$ s.t. $f=f^c$ or $\exists\, x,\ y\in[0,\ 1]$ s.t.

$$\frac{f(x)\cdot f(y)}{f(1-x)\cdot f(1-y)}\neq\frac{f\left(\dfrac{xy}{(1-x)(1-y)+xy}\right)}{f\left(\dfrac{(1-x)(1-y)}{(1-x)(1-y)+xy}\right)}.$$

Proof. We show that if

$$\frac{f(x)\cdot f(y)}{f(1-x)\cdot f(1-y)}=\frac{f\left(\dfrac{xy}{(1-x)(1-y)+xy}\right)}{f\left(\dfrac{(1-x)(1-y)}{(1-x)(1-y)+xy}\right)}\quad\forall\ x,y\in[0,1],\tag{3}$$

then $\exists c\in\mathbb{R}$ for which $f(x)=x^{1/c}/(x^{1/c}+(1-x)^{1/c})$. Define

$$h(s)=\frac{f\left(\dfrac{s}{1+s}\right)}{f\left(\dfrac{1}{1+s}\right)}.$$

It is easy to verify that

$$h(x/(1-x))=f(x)/f(1-x)\quad\forall\ x\ \in(0,\ 1).$$

Also note that

$$h\left(\frac{xy}{(1-x)(1-y)}\right)=\frac{f\left(\dfrac{xy}{(1-x)(1-y)+xy}\right)}{f\left(\dfrac{(1-x)(1-y)}{(1-x)(1-y)+xy}\right)}.\tag{4}$$

Define $q(t)=t/(1-t)$. By (3) and (4):

$$h(q(x))\cdot h(q(y))=h(q(x)\cdot q(y))\ \forall\ x,y\in[0,\ 1].\tag{5}$$

As $[0,\ 1]\subset q([0,\ 1])$, we conclude that $h(s)\cdot h(t)=h(s\cdot t)\ \forall\ s,t\in[0,\ 1]$. As h is continuous (remember that f is continuous), it is easily seen that $\exists\, d\in\mathbb{R}$ s.t. $h(s)=s^d$. Therefore, for all values of x we have $f(x)/f(1-x)=h(x/(1-x))=h(q(x))=q(x)^d=(x/(1-x))^d$. Combine this with $f(x)+f(1-x)=1$ to get $f(x)=x^d/x^d+(1-x)^d$. By the fact that $\lim_{x\to 0} f(x)=0$ we can rule out the case $d\leq 0$.

Denote $c=(1/d)$ to obtain $f(x)=x^{1/c}/x^{1/c}+(1-x)^{1/c}$ for $c>0$. □

Lemma 7. *If* $\phi:\mathscr{D}\to\mathbb{R}$ *satisfies* (*C*), (*CRA*), (*O*) *and* (*B*), *then* $\exists c\in\mathbb{R}$ *s.t.* $f_\phi=f^c$.

Proof. Suppose the claim of the lemma was incorrect, i.e., there exists a fair solution, ϕ, for which $f = f_\phi \neq f^c$ for any c. By Lemmas 4 and 5, there exist $x, y \in [0, 1)$ s.t.

$$\frac{f(x)\cdot f(y)}{f(1-x)\cdot f(1-y)} \neq \frac{f\left(\frac{xy}{(1-x)(1-y)+xy}\right)}{f\left(\frac{(1-x)(1-y)}{(1-x)(1-y)+xy}\right)}. \tag{6}$$

Obviously, $x \neq 1-y$. We now construct an example which will violate the *betweenness* axiom.

Take any three positive numbers – e, b, d – which satisfy, simultaneously, the following three equations:

$$\frac{e}{e+d} = x, \quad \frac{b}{b+e} = y, \quad 4e + b + d = 1.$$

Take $\alpha = 1/2f(y)$. Clearly, $\alpha > 1/2$. Take

$$\beta = \frac{\frac{1}{2} - f\left(\frac{b}{b+d}\right)\cdot\alpha}{1 - f\left(\frac{b}{b+d}\right)}. \tag{7}$$

As $y \neq 1-x$ we may assume without loss of generality (w.l.o.g.) that $y > 1-x$ and, therefore, $x > 1-y \Rightarrow e/(e+d) > e/(e+b) \Rightarrow d < b \Rightarrow b/(b+d) > 1/2 \Rightarrow f(b/(b+d)) \geq 1/2$.

Let X be the following random variable:

$$X = \begin{pmatrix} \beta & 0 & \alpha & 1 \\ d & 2e & b & 2e \end{pmatrix}.$$

X can be written as a convex combination of two random variables as follows:

$$X = (b+d)\cdot\begin{pmatrix} \beta & \alpha \\ \frac{d}{b+d} & \frac{b}{b+d} \end{pmatrix} + 4e\cdot\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \equiv (b+d)\cdot X_1 + 4eX_2. \tag{8}$$

Note that:

$$\phi(X_2) = 1/2, \tag{9}$$

and (by plugging in the definitions of α, β):

$$\begin{aligned}\phi(X_1) &= \beta + (\alpha - \beta)\cdot f\Big(\frac{b}{b+d}\Big)\\ &= \frac{\frac{1}{2} - f\Big(\frac{b}{b+d}\Big)\cdot\frac{1}{2f(y)}}{1 - f\Big(\frac{b}{b+d}\Big)}\\ &\quad + \left[\frac{1}{2f(y)} - \left(\frac{\frac{1}{2} - f\Big(\frac{b}{b+d}\Big)\cdot\frac{1}{2f(y)}}{1 - f\Big(\frac{b}{b+d}\Big)}\right)\right] f\Big(\frac{b}{b+d}\Big)\\ &= \frac{f(y) - f\Big(\frac{b}{b+d}\Big)}{2f(y)\Big(1 - f\Big(\frac{b}{b+d}\Big)\Big)} + \frac{1 - f(y)}{2f(y)\Big(1 - f\Big(\frac{b}{b+d}\Big)\Big)}\cdot f\Big(\frac{b}{b+d}\Big) = \frac{1}{2}.\end{aligned} \tag{10}$$

So, by (8), (9) and (10),

$$\phi(X) = 1/2. \tag{11}$$

We now look at X as a convex combination of three other lotteries:

$$\begin{aligned}X &= 2e\cdot\begin{pmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix} + (b+e)\begin{pmatrix}0 & \alpha\\ \frac{e}{b+e} & \frac{b}{b+e}\end{pmatrix} + (e+d)\begin{pmatrix}\beta & 1\\ \frac{d}{e+d} & \frac{e}{e+d}\end{pmatrix}\\ &\equiv 2e\cdot X_3 + (b+e)\cdot X_4 + (e+d)\cdot X_5.\end{aligned} \tag{12}$$

X_3 and X_4 satisfy:

$$\phi(X_3) = \frac{1}{2} \text{ and } \phi(X_4) = \alpha\cdot f\left(\frac{b}{b+e}\right) = \frac{1}{2f(y)}\cdot f(y) = \frac{1}{2}\,. \tag{13}$$

We conclude by (11), (12) and (13) that $\phi(X_5) = 1/2$. We show that this is impossible.
Suppose

$$\frac{1}{2} = \phi(X_5) = \beta + (1-\beta)\cdot f\left(\frac{e}{e+d}\right).$$

So,

$$f(x) = f\Big(\frac{e}{e+d}\Big) = \frac{\frac{1}{2} - \beta}{1-\beta} = \frac{1-2\beta}{2-2\beta} \tag{14}$$

$$f(y) = f\Big(\frac{b}{e+b}\Big) = \frac{1}{2\alpha} \tag{15}$$

$$\Rightarrow \frac{f(x)f(y)}{(1-f(x))(1-f(y))} = \frac{\dfrac{1-2\beta}{2-2\beta}\cdot\dfrac{1}{2\alpha}}{\dfrac{1}{2-2\beta}\cdot\dfrac{2\alpha-1}{2\alpha}} = \frac{1-2\beta}{2\alpha-1}\,. \tag{16}$$

On the other hand,

$$\frac{xy}{(1-x)(1-y)+xy} = \frac{b}{b+d} \tag{17}$$

which implies, by (7):

$$\frac{f\left(\dfrac{xy}{(1-x)(1-y)+xy}\right)}{f\left(\dfrac{(1-x)(1-y)}{(1-x)(1-y)+xy}\right)} = \frac{f\left(\dfrac{b}{b+d}\right)}{f\left(\dfrac{d}{b+d}\right)} = \frac{\dfrac{\frac{1}{2}-\beta}{\alpha-\beta}}{\dfrac{\alpha-\frac{1}{2}}{\alpha-\beta}} = \frac{1-2\beta}{2\alpha-1}. \tag{18}$$

Now by (16) and (18) we obtain:

$$\frac{f(x)\,\cdot\,f(y)}{f(1-x)\cdot f(1-y)} = \frac{f\left(\dfrac{xy}{(1-x)(1-y)+xy}\right)}{f\left(\dfrac{(1-x)\,(1-y)}{(1-x)(1-y)+xy}\right)}.$$

which contradicts (6). □

So far, we have shown that if ϕ is a fair solution, its underlying function must coincide with an underlying function for some ϕ_c. The following lemma is, therefore, the last step in the proof of the second direction in Theorem A:

Lemma 8. *If α and ϕ are two fair solutions satisfying $f_\alpha \equiv f_\phi$ then $\alpha \equiv \phi$.*

Proof. By (C) it is sufficient to show that if α and ϕ satisfy $f_\alpha = f_\phi$ then they coincide on any atomic random variable with finitely many atoms. Obviously, this is true for any Bernoulli random variable. The extension to random variables with two atoms is straightforward. By invariance, f_α defines α on any atomic random variable having two atoms:

$$\alpha\begin{pmatrix} a & b \\ 1-p & p \end{pmatrix} = a + (b-a)\cdot f_\alpha(p) = a + (b-a)\cdot f_\phi(p) = \phi\begin{pmatrix} a & b \\ 1-p & p \end{pmatrix}.$$

We proceed by induction. Assume the claim is true for any atomic random variable with n atoms. Assume there exists a random variable, X, with $n+1$ atoms such that $\alpha(X) \neq \phi(X)$. Obviously, X can be written as a convex combination of two random variables X_1, X_2, with 2 and n atoms, correspondingly, such that $\alpha(X_1) = \alpha(X_2) = \alpha(X)$. But in that case, the induction hypothesis implies $\phi(X_1) = \alpha(X_1) = \alpha(X_2) = \phi(X_2)$, and by (B) $\phi(X) = \phi(X_1) = \alpha(X_1) = \alpha(X)$. Contradiction. □

Lemmas 7 and 8 provide the second direction in the proof of the main result. Namely, all fair solutions are of the form ϕ_c. □

4. Concluding remarks

4.1. Betweenness and Gâteaux differentiability

The assumption on the differentiability of certainty equivalents plays an important role in expected utility and nonexpected utility theory. For example, Machina (1982) restricts attention to functions which are assumed, at the outset, to be Fréchet differentiable. Later papers consider axiomatizations which lead to Gâteaux differentiability, a strictly weaker form of differentiability (e.g., Chew et al., 1987). A natural question which therefore arises concerns sufficient axioms over preferences for Gâteaux differentiability. The functionals introduced here ($\phi_c(X) = \text{argmin}_t E(|X - t|^{c+1})$ for $c > 0$ and $c \backslash\text{not} = 1$) are not Gâteaux (and, therefore, not Fréchet) differentiable. This demonstrates that (C), (CRA), (O) and even (B) are not enough to ensure differentiability.[4]

4.2. The reflection effect

Modeling the reflection effect and computing certainty equivalents for lotteries which satisfy this effect is interesting not only in the context of an individual's preference relation. Another instance, for example, where the reflection effect is natural is when a decision maker has to decide on the value of a lottery before actually knowing whether he will eventually pay or be paid the lottery. Think for example of the following variant of the cake division problem. Two partners of a joint venture must terminate their partnership, and one partner would like to buy out the other. The joint venture might be modeled as a random future income (lottery) and a fair process to pursue is for one partner to announce a value (certainty equivalent) for the business and for the other partner to decide whether he will buy or sell his share, at the announced value. The value announced by the first partner should naturally comply with the reflection effect.

Another instance where the introduction of the reflection effect is natural, and where it may actually be viewed as a desired norm, is for the design of social arbitration institutions. Consider any such institution designed to settle bilateral conflicts. Assume that true justice would lead to a payment, X, of one party to another, where X is a random variable from the perspective of a third party. This third party (the arbitration institution) must replace the random variable X by a fixed sum. Obviously, the decisions of this institution should satisfy some anonymity property, where the names and identities of parties to a dispute should not matter for the process of the dispute resolution. Note that anonymity translates into the reflection effect. Thus, one may view Theorem A as normative in this context.

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[4]This was pointed out in Safra and Segal (1998).

4.3. Application to the bargaining problem

In Nash's bargaining problem (Nash, 1950) agents are assumes to maximize expected utility and the disagreement point is therefore assumed to be constant. Nash's solution to this problem is the unique solution concept to comply with a set of four plausible axioms. Smorodinsky (1995) studies Nash's problem in a setup where agents satisfy weaker axioms. Specifically he extends Nash's solution to the case of a random disagreement point. The main result of this paper plays a major role for proving the uniqueness of the solution concept introduced there.

4.4. Empirical evidence

Karmarkar (1978) studies *subjectively weighted utilities*, where the weights of each outcome in a lottery are not the probabilities. Rather, they are derived subjectively from the probability distribution. Looking at empirical evidence, Karmarkar provides a representation theorem, introducing functionals which bear some resemblance with results obtained here.[5]

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