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The speed of rational learning*

Alvaro Sandroni[1], **Rann Smorodinsky**[1,2]

[1] J. L. Kellogg Graduate School of Management, MEDS Department, Northwestern University 2001 Sheridan Road, Evanston, IL 60208, USA (e-mail: sandroni@nwu.edu)
[2] Faculty of Industrial Engineering and Management, Technion, Haifa 32000, Israel (e-mail: rann@ie.technion.ac.il)

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Abstract. A central result in the rational learning literature is that if the true measure is absolutely continuous with respect to the beliefs then, given enough data, the updated beliefs merge with the true distribution. In this paper, we show that, under absolute continuity, weak merging occurs fast (at the rate $1/\sqrt{t}$) with density one. Moreover, if weak merging occurs fast enough (at the rate $1/t$) then absolute continuity holds. These rates are sharp. We also show that, under some conditions, if weak merging occurs at the rate $1/\sqrt{t}$ then absolute continuity holds.

Key words: Rational learning, merging, speed of convergence

1. Introduction

A central result in the rational learning literature is that, given enough data, updated beliefs merge with the true measure if and only if the true measure is absolutely continuous with respect to the beliefs (see Blackwell and Dubins (1962) and Kalai and Lehrer (1994)). Hence, under absolute continuity, convergence to Nash equilibrium obtains (see Kalai and Lehrer (1993a)). Absolute continuity requires that if an event has positive probability under the true probability measure then it has positive probability under the beliefs.

The central question in this paper is the speed of the convergence process under absolute continuity. In game theoretical and economic models, the literature on this issue is small. Jordan (1992) obtained an exponential rate of

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convergence to Nash equilibrium for a class of myopic learning processes. Vives (1993) showed interesting examples of slow convergence (at the rate $t^{-1/6}$) and fast convergence (at the rate $t^{-0.5}$) in an economic model.

There are two standard notions of convergence. One is merging and the other is weak merging. The beliefs merge with the true measure if the predictions about all future events eventually become accurate. The beliefs weakly merge with the true measure if eventually the predictions about all future events become accurate, except, possibly, distant-future events. Although weak merging is weaker than merging, Lehrer and Smorodinsky (1997) and Sandroni (1998) have shown that weak merging is a sufficient condition for convergence to Nash equilibrium.

The central results in this paper are as follows:

1. If the true measure is absolutely continuous with respect to the beliefs then the true measure and the beliefs weakly merge at the rate $t^{-0.5}$ in a subsequence of periods which has density one.
2. If the beliefs and the true measure weakly merge at the rate $t^{-(1+\varepsilon)}$, for an arbitrary $\varepsilon > 0$, then the true measure is absolutely continuous with respect to the beliefs.
3. If the beliefs over next period's outcomes are uniformly bounded away from zero and the beliefs and the true measure weakly merge at the rate $t^{-(0.5+\varepsilon)}$, for an arbitrary $\varepsilon > 0$, then the true measure is absolutely continuous with respect to the beliefs.

All three results are tight.

The paper is organized as follows: In section 2, we present an example illustrating the connection between absolute continuity and fast weak merging. In section 3, the basic set-up is defined. In section 4, the different notions of convergence are defined. Our results hinge on a characterization of absolute continuity by Kabanov, Liptser and Shiryaev (1977) revisited in section 5. The main results are presented in section 6. Section 7 contains applications to the theory of rational learning in repeated games.

2. Motivating example

Assume that there are two outcomes: heads and tails. The true probability of heads is 1 in all periods. At period t, the probability of heads is $1 - \frac{1}{(t+2)^{\rho}}$ under the belief. If $\rho > 0$ then the belief and true measure weakly merge. However, the higher is ρ, the faster is the convergence rate. Absolute continuity (and merging) obtains if only if $\rho > 1$. This is so because absolute continuity holds if and only if the probability of the event "heads in all periods" is strictly positive under the belief. The logarithmic of this probability is $\sum_{j=1}^{\infty} \log\left(1 - \frac{1}{(j+2)^{\rho}}\right) = -\sum_{j=1}^{\infty} \frac{1}{\varepsilon} \frac{1}{(j+2)^{\rho}}$ where $1 - \frac{1}{(j+2)^{\rho}} \leq \varepsilon \leq 1$. Hence, under the belief, the logarithmic of the probability of "heads in all periods" is minus infinity if and only $\rho \leq 1$.

The focus of this paper is to establish the connection between merging and the rate of weak merging for arbitrary probability measures.

3. The basic set-up

Let $\sum$ be a finite set. Let $\sum^t$ be the set of $(t+1)$-histories, $0 \le t \le \infty$. Let $H = \bigcup_{0 \le t < \infty} \sum^t$ be the set of finite histories. A cylinder with base on $h \in \sum^t$ is the set $C(h) = \{w \in \sum^\infty \mid w = (h, \ldots)\}$ of all infinite histories such that the $t+1$ initial elements coincide with h. Let $\Im_0 \subset \cdots \Im_t \subset \cdots \subset \Im$, be a filtration where $\Im_0$ is the trivial σ-algebra, $\Im_t$ is the σ-algebra generated by the cylinders with base on $\sum^t$, and $\Im$ is the σ-algebra generated by the algebra $\Im^0 \equiv \bigcup_{t \ge 0} \Im_t$.

Let μ and $\tilde{\mu}$ be two probability measures on $(\sum^\infty, \Im)$. Let μ be called "the true probability measure" and let $\tilde{\mu}$ be called "the belief." After history $h \in H$, the posterior of the belief and the true measure are given by μ_h and $\tilde{\mu}_h$, respectively.

4. Notions of convergence

After history $h \in H$, the difference in the sup-norm between the belief and the true measure is $\|\mu_h - \tilde{\mu}_h\| \equiv \sup_{A \in \Im} |\mu_h(A) - \tilde{\mu}_h(A)|$. The belief and the true measure merge if $\|\mu_h - \tilde{\mu}_h\| \xrightarrow[t \to \infty]{} 0(\mu\ a.s.)$. That is, the belief and the true measure merge if, given enough data, the probabilities assigned by the updated belief and true measure become arbitrarily close.

After history $h \in H$, the difference in the d_l-metric between the updated belief and true measure is $d_l(\mu_h, \tilde{\mu}_h) \equiv \sup_{A \in \Im_j, 0 \le j \le l} |\mu_h(A) - \tilde{\mu}_h(A)|$. The belief and the true measure weakly merge if, for every natural number l, $d_l(\mu_h, \tilde{\mu}_h) \xrightarrow[t \to \infty]{} 0(\mu\ a.s.)$. That is, the belief and the true measure weakly merge if, given enough data, the probabilities assigned by the updated belief and true measure to events within finitely many periods become arbitrarily close.

Definition 1. *The belief and the true measure weakly merge at the rate t^{-v} if, for every natural number l, μ a.s., $t^v d_l(\mu_h, \tilde{\mu}_h)$ converges to zero.*

That is, the belief and the true measure weakly merge at the rate t^{-v} if the d_l-distance between the belief and the true measure goes to zero faster than t^{-v}.

The density of a subsequence $L \subseteq N$ is defined as $\limsup_{m \to \infty} \#\{L \cap \{1, \ldots, m\}/m$. A sequence a_t goes to zero with density one if for every $\varepsilon > 0$, $\dfrac{\#\{j/|a_j| \le \varepsilon; j \le t\}}{t} \xrightarrow[t \to \infty]{} 1$. That is, a sequence a_t goes to zero with density one if a_t becomes arbitrarily small in a subsequence of density one.

Definition 2. *The belief and the true measure weakly merge with density one at the rate t^{-v} if, for every natural number l, μ a.s., $t^v d_l(\mu_h, \tilde{\mu}_h)$ goes to zero, with density one.*

That is, the belief and true measure weakly merge with density one at the rate t^{-v} if the d_l-distance between the updated belief and true measure goes to zero with density one faster than $t^{-\nu}$. Naturally, in a subsequence of density zero, the belief and the true measure may weakly merge at a slower rate or not at all. The standard rate for fast convergence is $t^{-0.5}$ (see Vives (1993)).

Definition 3. *The belief and the true measure weakly merge fast with density one if the belief and the true measure weakly merge with density one at the rate* $t^{-0.5}$.

5. A characterization of absolute continuity

In this section, absolute continuity is formally defined. Moreover, it is shown a characterization of absolute continuity (due to Kabanov, Liptser and Shiryaev) which will be useful to prove the main results of this paper and to determine when absolute continuity (and merging) occurs in specific examples.

μ is locally absolutely continuous with respect to $\tilde{\mu}$ if for every $A \in \Im^0$, $\tilde{\mu}(A) = 0$ implies $\mu(A) = 0$. μ is absolutely continuous with respect to $\tilde{\mu}$ if for every $A \in \Im$, $\tilde{\mu}(A) = 0$ implies $\mu(A) = 0$.

The assumption of local absolute continuity requires that any finite-time event which has zero probability under the belief also has zero probability under the true measure. Hereafter, we assume that the true measure is locally absolutely continuous with respect to the belief. This is a mild assumption which allows the belief to be unambiguously updated by Bayes' rule.

The assumption of absolute continuity requires that any event which has zero probability under the belief also has zero probability under the true measure. Absolute continuity is a much stronger condition than local absolute continuity.

The question that will now be considered is the conditions under which absolute continuity (and, consequently, merging) holds.

For every $x \in \Re$, $u(x) = x$ if $|x| \le 1$ and $u(x) = sign(x)$ if $|x| > 1$. Given $w \in \Sigma^\infty$, $w = (w(t), \ldots)$, $w(t) = (w(t-1), a)$, $a \in \Sigma$, let z_t be the $\Im_t$-measurable functions defined by:

$$z_t(w) = 0 \text{ if } \mu(C(w(t))) = 0; \quad \text{and}$$

$$z_t(w) = \log \frac{\tilde{\mu}(C(w(t)))}{\mu(C(w(t)))} \text{ if } \mu(C(w(t))) > 0.$$

That is, z_t is the logarithm of the ratio of the belief to the true measure. Let e_t be $E\{u(z_t) \mid \Im_{t-1}\}$ and let v_t be $E\{(u(z_t))^2 \mid \Im_{t-1}\}$, where E is the expectation operator associated with the true measure (e_t is non-positive, Shiryaev (1991) page 529).

Proposition 1. *The true measure is absolutely continuous with respect to the belief if and only if*

$$\sum_t e_t > -\infty \quad \text{and} \quad \sum_t v_t < \infty (\mu \ a.s.).$$

Proof: See Kabanov, Liptser and Shiryaev (1977), theorem 2, or Shiryaev (1991) page 530, equation (26).

6. Main results

The main results of this paper (propositions 2, 3 and 4) are presented below.

Proposition 2. *If the true measure is absolutely continuous with respect to the belief then the belief and the true measure weakly merge fast, with density one.*

Proof: See Appendix.

Corollary 1. *If the belief and the true measure merge then the belief and the true measure weakly merge fast, with density one.*

So, absolute continuity (or merging) implies fast weak merging, except in some rare periods.

Proposition 3. *If the belief and the true measure weakly merge at the rate $t^{-(1+\varepsilon)}$, for some $\varepsilon > 0$, then the true measure is absolutely continuous with respect to the belief. Therefore, the belief and the true measure merge.*

Proof: See Appendix.

So, absolute continuity (and merging) are necessary conditions for weak merging at the rate $t^{-(1+\varepsilon)}$, $\varepsilon > 0$.

Definition 4. *The beliefs over next period's outcomes are uniformly bounded away from zero if there exists $\gamma > 0$ such that $\tilde{\mu}_h(C(h, a)) > \gamma$ for every outcome $a \in \Sigma$, and every finite-history $h \in H$ such that $\tilde{\mu}(C(h)) > 0$.*

That is, the beliefs over next period's outcomes are uniformly bounded if there is $\gamma > 0$ such that the conditional probabilities (of $\tilde{\mu}$) over next period's outcomes are above γ.

Proposition 4. *Assume that the beliefs over next period's outcomes are uniformly bounded away from zero. If the belief and the true measure weakly* merge at the rate $t^{-(0.5+\varepsilon)}$, *for some $\varepsilon > 0$, then the true measure is absolutely continuous with respect to the belief. Therefore, the belief and the true measure merge.*

Proof: See Appendix.

So, under the assumption that the beliefs over next period's outcomes are uniformly bounded away from zero, absolute continuity (and merging) are necessary conditions for weak merging at the rate $t^{-(0.5+\varepsilon)}$, $\varepsilon > 0$.

Proposition 2 shows that, under absolute continuity, the belief and the true measure weakly merge fast with density one. Proposition 3 and 4 shows that if the belief and the true measure weakly merge fast enough then absolute continuity holds. Propositions 3 and 4 are not the converse of proposition 2. Hence, a natural question is whether it is possible to relax the assumptions in proposition 3 and 4 or to strengthen the conclusions in proposition 2. For example, in propositions 3 and 4, is it possible to assume that weak merging occurs with density one? Is it possible to assume slower rates of convergence? Similarly, in proposition 2, is it possible to obtain faster rates of convergence with density one? Is it possible to obtain fast weak merging, instead of fast weak merging with density one? The answer to these questions are no, as demonstrated by the examples below.

Example 1. It is not possible to dispose the proviso "with density one" in proposition 2.

Assume that the true probability of heads is 0.5 in every period. In all periods t such that there exists a natural number $j > 1$ such that $t = j^4$, the belief assigns probability $0.5(1 + 1/t^{0.25})$ to heads. In all other periods, the belief assigns probability 0.5 to heads. The subsequence $L = \{t \mid t = j^4, j \in N\}$ has density zero. For all paths $w = (w(t), \ldots)$, $\sqrt{t} d_1(\mu_{w(t)}, \tilde{\mu}_{w(t)}) = 0.5t^{0.25}$, $t \in L$. Hence, the belief and the true measure do not weakly merge at the rate $t^{-0.5}$ in this subsequence. By definition, if t is large enough, $e_t = 0$ if $t \neq j^4$, and $e_t = 0.5 \log\left(1 - \frac{1}{j^2}\right)$ if $t = j^4$; $v_t = 0$ if $t \neq j^4$, and $v_t = 0.5 \log^2\left(1 - \frac{1}{j}\right) + 0.5 \log^2\left(1 + \frac{1}{j}\right)$ if $t = j^4$. However, if $x \geq 0$ then $\log(1 + x) \leq x$; and if $x < 0$, but sufficiently close to zero, then $\log(1 - x) \geq (-2)x$. Hence, if j is sufficiently large then $0 \geq e_t \geq -\frac{1}{j^2}$, and $0 \leq v_t \leq \frac{2.5}{j^2}$. Hence, by proposition 1, the true measure is absolutely continuous with respect to the belief.

Example 2. It is not possible to obtain faster rates of convergence in proposition 2 (from $t^{-0.5}$ to t^{-v}, $v > 0.5$).

Assume, as in the previous example, that the true probability of heads is 0.5 in every period. The belief is that the probability of heads is $0.5\left(1 + \frac{1}{t^\gamma}\right), \gamma > 0.5$, in all periods. If $v > \gamma > 0.5$ then, for all paths $w = (w(t), \ldots)$, $t^v d_1(\mu_{w(t)}, \tilde{\mu}_{w(t)})$ goes to infinity as t goes to infinity. So, the belief and the true measure do not weakly merge, with density one, at the rate t^{-v}. By definition, if t is sufficiently large, $e_t = 0.5 \log\left(1 - \frac{2}{t^{2\gamma}}\right)$, and $v_t = 0.5 \log^2\left(1 - \frac{2}{t^\gamma}\right) + 0.5 \log^2\left(1 + \frac{2}{t^\gamma}\right)$. Hence, if t is sufficiently large then $0 \geq e_t \geq -\frac{4}{t^{2\gamma}}$ and $0 \leq v_t \leq \frac{10}{t^{2\gamma}}$. By proposition 1, the true measure is absolutely continuous with respect to the belief.

Example 3. It is not possible to assume slower rates of convergence in proposition 3 (from $t^{-(1+\varepsilon)}$, $\varepsilon > 0$, to t^{-1}).

Assume that, as in the motivating example, the true probability of heads is

1 in every period. At period $t \geq 2$, the belief assigns probability $1 - \frac{1}{t \log t}$ to heads next period. Then, the belief and the true measure weakly merge at rate t^{-1}. If t is sufficiently large, $e_t = \log\left(1 - \frac{1}{t \log t}\right)$. However, $\log(1 - x) \leq -x$. So, $e_t \leq -\frac{1}{t \log t}$. Moreover, $\sum_{t=2}^{\infty} \frac{1}{t \log t} = \infty$ (this equality is a direct consequence of the Cauchy condensation test, (see Goldberg (1976), page 88, theorem 3.7b). By proposition 2, the true measure is not absolutely continuous with respect to the belief.

Example 4. It is not possible to assume slower rates of convergence in proposition 4 (from $t^{-(0.5+\varepsilon)}$, $\varepsilon > 0$, to $t^{-0.5}$).

Assume that the true probability of heads is 0.5 in every period. At period $t \geq 2$, the belief assigns probability $0.5\left(1 + \frac{1}{\sqrt{t \log t}}\right)$ to heads next period. Then, the belief and the true measure weakly merge at the rate $t^{-0.5}$ and the beliefs over next period's outcomes are uniformly bounded away from zero. If t is sufficiently large, $e_t = 0.5 \log\left(1 - \frac{1}{t \log t}\right)$. By the same argument given in the previous example, the true measure is not absolutely continuous with respect to the belief.

Example 5. It is not possible to assume that weak merging occurs with density one in propositions 3 and 4.

Assume, as in the previous example, that the true probability of heads is 0.5 in every period. The belief assigns probability 0.3 to "heads" in all periods t such that $t = j^2$ for some natural number $j > 1$, and 0.5 to "heads" in all other periods. Then, the belief and the true measure weakly merge at any rate t^{-v}, $v > 0$, with density one, but the belief and the true measure do not merge (nor weakly merge).

7. An application to rational learning in game theory

In this section, a finite number of long-lived players play an infinitely repeated game. In each period, players' choose an action and update their beliefs about the future evolution of the play according to the past history of outcomes. The model is formally described below.

Using the notation from before, assume that each player i has a finite set $\sum_i$ of possible actions. Let $\Delta(\sum_i)$ be the set of probability distributions on $\sum_i$. Let $\sum = \prod_{i=1}^{n} \sum_i$ be the set of action combinations.

Each player i has a payoff function $u_i : \sum \to \Re$; a chosen behavior strategy $f_i : H \to \Delta(\sum_i)$ which describes how player i randomizes among possible actions conditional on every possible history; and a belief about opponents' strategies $f^i = \left(f_1^i, \ldots, f_n^i\right)$. The chosen strategy profile is defined by $f = (f_1, \ldots, f_n)$. Each player knows his own strategy, i.e., $f_i = f_i^i$.

Given a strategy profile g, there exists a probability measure μ_g (see Kalai and Lehrer (1993a) for details) that represents the probability distribution over play paths generated by g. Let μ_f and μ_{f^i} be the probability measures associated with the strategy profiles f and f^i, respectively. That is, μ_f is the true probability distribution of the play and μ_{f^i} is player i's belief about the evolution of the play.

Propositions 2, 3, and 4 can be immediately applied, replacing μ with μ_f and $\tilde{\mu}$ with μ_{f^i}. Assume that the true measure is absolutely continuous with respect to player i's belief, then player i's belief and the true measure weakly merge fast with density one. Hence, if, for every player i, the true measure is absolutely continuous with respect to the player i's beliefs, then there is convergence to an equilibrium, in a sequence of density one, at the rate $t^{-0.5}$. Moreover, if player i's belief and the true measure weakly merge at the rate $t^{-(1+\varepsilon)}$, for $\varepsilon > 0$, then the true measure is absolutely continuous with respect to the belief. Hence, absolute continuity and merging are necessary conditions for weak merging at the rate $t^{-(1+\varepsilon)}$, for $\varepsilon > 0$. Furthermore, if, players believe that the probabilities of next period's outcomes are bounded away from zero then absolute continuity and merging are necessary conditions for weak merging at the rate $t^{-(0.5+\varepsilon)}$, for $\varepsilon > 0$.

8. Appendix

Lemma A.1.

1. $xu(\log x) \geq x - 1$ *if* $x \geq 0$;
2. $u(\log x) \leq (x - 1)$ *if* $x \geq 1$;
3. $u(\log x) \geq \frac{e}{e-1}(x - 1)$ *if* $x \leq 1$;
4. $|u(\log x)| \leq \frac{e}{e-1}|x - 1|$.

Proof: For part 1, see Shiryaev (1991) page 529. Part 2 follows from $log(x) \leq x - 1$. Part 3 holds when $x \leq \frac{1}{e}$ because then $\frac{e}{e-1}(x-1) \leq -1$. The inequality in part 3 holds as an equality when $x = 1$. So, the inequality holds for all value between $\frac{1}{e}$ and 1 because $\log x$ is a concave function of x. Part 4 follows from parts 2 and 3. *q.e.d.*

Lemma A.2. *Let* $\{a_i, i = 1, \ldots, T\}$ *and* $\{b_i, i = 1, \ldots, T\}$ *be two finite sequences of real numbers such that* $a_i \geq 0$, $b_i \geq 0$, (*if* $b_i = 0$ *then* $a_i = 0$), $\sum_i a_i = 1$, *and* $\sum_i b_i = 1$. *Then,*

1. $\sum_i a_i \left(u\left(\log \frac{b_i}{a_i} \right) \right)^2 \geq \left(\sum_{i:a_i \geq b_i > 0} (a_i - b_i) \right)^2.$
2. $\sum_i a_i u\left(\log \frac{b_i}{a_i} \right) \geq -\frac{e}{e-1} \sum_{i:a_i \geq b_i > 0} (a_i - b_i).$
3. $\sum_i a_i \left(u\left(\log \frac{b_i}{a_i} \right) \right)^2 \leq \frac{2e}{e-1} \left(\sum_{i:a_i \geq b_i > 0} (a_i - b_i) \right).$

Proof: Part 1. Denote $A \equiv \sum_{i:a_i \geq b_i > 0} a_i$. Clearly, $A \leq 1$.

$$\sum_i a_i \left(u \left(\log \frac{b_i}{a_i} \right) \right)^2 = \sum_i a_i \left(u \left(\log \frac{a_i}{b_i} \right) \right)^2 \geq \sum_{i:a_i \geq b_i > 0} a_i \left(u \left(\log \frac{a_i}{b_i} \right) \right)^2$$

$$= A \sum_{i:a_i \geq b_i > 0} \frac{a_i}{A} \left(u \left(\log \frac{a_i}{b_i} \right) \right)^2$$

$$\geq A \left(\sum_{i:a_i \geq b_i > 0} \frac{a_i}{A} u \left(\log \frac{a_i}{b_i} \right) \right)^2$$

$$= \frac{1}{A} \left(\sum_{i:a_i \geq b_i > 0} b_i \frac{a_i}{b_i} u \left(\log \frac{a_i}{b_i} \right) \right)^2$$

$$\geq \left(\sum_{i:a_i \geq b_i > 0} b_i \frac{a_i}{b_i} u \left(\log \frac{a_i}{b_i} \right) \right)^2.$$

Since $xu(log(x)) \geq x - 1$ for $x \geq 0$,

$$\sum_{i:a_i \geq b_i > 0} b_i \frac{a_i}{b_i} u \left(\log \frac{a_i}{b_i} \right) \geq \sum_{i:a_i \geq b_i > 0} b_i \left(\frac{a_i}{b_i} - 1 \right) = \sum_{i:a_i \geq b_i > 0} (a_i - b_i).$$

Hence,

$$\left(\sum_{i:a_i \geq b_i > 0} b_i \frac{a_i}{b_i} u \left(\log \frac{a_i}{b_i} \right) \right)^2 \geq \left(\sum_{i:a_i \geq b_i > 0} (a_i - b_i) \right)^2.$$

This concludes part 1.

Part 2. Since $u(\log x) \geq \frac{e}{e-1}(x-1)$ if $x \leq 1$ (see lemma A.1),

$$\sum_i a_i u \left(\log \frac{b_i}{a_i} \right) \geq \sum_{i:a_i \geq b_i > 0} a_i u \left(\log \frac{b_i}{a_i} \right) \geq \sum_{i:a_i \geq b_i > 0} a_i \frac{e}{e-1} \left(\frac{b_i}{a_i} - 1 \right)$$

$$= \sum_{i:a_i \geq b_i > 0} \frac{e}{e-1} (b_i - a_i) = -\frac{e}{e-1} \sum_{i:a_i \geq b_i > 0} (a_i - b_i).$$

This concludes part 2.

Part 3. Assume that $a_i > 0$ and $b_i > 0$. Then,

$$\left(u \left(\log \frac{b_i}{a_i} \right) \right)^2 \leq \left| u \left(\log \frac{b_i}{a_i} \right) \right| \leq \frac{e}{e-1} \left| \frac{b_i}{a_i} - 1 \right|$$

because $\left| u \left(\log \frac{b_i}{a_i} \right) \right| \leq 1$ and because $|u(\log x)| \leq \frac{e}{e-1} |x - 1|$ (see lemma

$A.1$). Therefore,

$$\sum_i a_i\left(u\left(\log\frac{b_i}{a_i}\right)\right)^2 = \sum_{i:a_i>0,b_i>0} a_i\left(u\left(\log\frac{b_i}{a_i}\right)\right)^2 \leq \frac{e}{e-1}\sum_{i:a_i>0,b_i>0} a_i\left|\frac{b_i}{a_i}-1\right|$$

$$= \frac{e}{e-1}\sum_{i:a_i>0,b_i>0} |b_i - a_i| \leq \frac{e}{e-1}\sum_{i:b_i>0} |b_i - a_i|$$

$$\leq \frac{2e}{e-1}\sum_{i:a_i \geq b_i>0} (a_i - b_i).$$

The last inequality comes from

$$\sum_{i:a_i \geq b_i>0} (a_i - b_i) = \sum_{i:a_i<b_i,b_i>0} (b_i - a_i). \qquad q.e.d.$$

Lemma A.3. *Consider a sequence* $\{c_n, n \geq 0\}$ *such that* $c_n > 0$ *and* $\sum_{n \geq 0} c_n < \infty$. *Then,* nc_n *converges to zero with density one.*

Proof: By Kronecker's lemma, it follows that $\frac{\sum_{j=0}^n jc_j}{n}$ converges to zero. Hence, if jc_j is greater than $\delta > 0$ in a subsequence of density $\beta > 0$ then $\limsup \frac{\sum_{j=0}^n jc_j}{n}$ is greater than $\beta\delta/2 > 0$. A contradiction. *q.e.d.*

Lemma A.4. *Fix* $\varepsilon > 0$. *If* $n^{(1+\varepsilon)}|c_n|$ *goes to zero then* $|\sum_{n \geq 0} c_n| < \infty$.

Proof: For n large enough, $|c_n| \leq \frac{1}{n^{(1+\varepsilon)}}$. So, $\sum_{n \geq 0} |c_n| < \infty$. However, for every natural number k, $|\sum_{n=1}^k c_n| \leq \sum_{n=1}^k |c_n|$. *q.e.d.*

Proof of Proposition 2: By proposition 1,

$$\sum_{t=0}^{\infty} v_t < \infty (\mu\ a.s.).$$

By lemma $A.3$, almost surely with respect to μ, tv_t converges to zero, with density one.

let η_t be $\Im_t$-measurable functions defined by:

$$\eta_t(w) = d_1(\tilde{\mu}_{w(t)}, \mu_{w(t)})$$

where $w \in \Sigma^{\infty}, w = (w(t), \ldots)$. By lemma $A.2$ (part 1),

$$0 \leq (\eta_t)^2 \leq v_t.$$

Thus, μ *a.s.*, $t^{0.5}d_1(\mu_{w(t)}, \tilde{\mu}_{w(t)})$ goes to zero, with density one. The proof that μ *a.s.*, $t^{0.5}d_l(\mu_{w(t)}, \tilde{\mu}_{w(t)})$ goes to zero, with density one, $l \geq 2$, is completely analogous and, therefore, is omitted. *q.e.d.*

Proof of Proposition 3: Let η_t be defined as in the proof of proposition 2. By lemma $A.2$ (part 2),

$$e_t \geq -\frac{e}{e-1}\eta_t.$$

By assumption, μ *a.s.*, $t^{(1+\varepsilon)}\eta_t$ goes to zero, with density one, for some $\varepsilon > 0$. By lemma $A.4$, μ *a.s.*,

$$\sum_{t=0}^{\infty} \eta_t < \infty.$$

Therefore, μ *a.s.*,

$$\sum_{t=0}^{\infty} e_t > -\infty.$$

By lemma $A.2$ (part 3),

$$v_t \leq \frac{2e}{e-1}\eta_t.$$

Hence,

$$\sum_{t=0}^{\infty} v_t < \infty.$$

By proposition 1, the true measure is absolutely continuous with respect to the belief. Hence, the belief and the true measure merge. *q.e.d.*

Let $\rho_t(w)$ be $\dfrac{\mu(C(w(t)))}{\tilde{\mu}(C(w(t)))}$ if $\tilde{\mu}(C(w(t))) > 0$ and 0 otherwise, $w \in \Sigma^{\infty}$, $w = (w(t), \ldots)$.

Lemma A.5. *Assume that the beliefs over next period's outcomes are uniformly bounded away from zero. Also assume that μ a.s.,*

$$\sum_{t=0}^{\infty} \tilde{E}\{(\rho_{t+1} - 1)^2 \mid \Im_t\} < \infty.$$

where $\tilde{E}$ is the expectation operator associated with $\tilde{\mu}$. Then, the true measure is absolutely continuous with respect to the belief.

Proof: See Shiryaev (1991), page 531, corollary 4.

Proof of Proposition 4: It is easy to see that

$$\tilde{E}\{(\rho_{t+1} - 1)^2 \mid \Im_t\} \leq \frac{1}{\gamma}(\eta_t)^2.$$

By assumption, there exists $\varepsilon > 0$ such that μ *a.s.*, $t^{-0.5+\varepsilon}\eta_t$ goes to zero. So, μ *a.s.*, $t^{-1+2\varepsilon}(\eta_t)^2$ goes to zero. By lemma $A.4$, μ *a.s.*,

$$\sum_{t=0}^{\infty}(\eta_t)^2 < \infty \Rightarrow \sum_{t=0}^{\infty}\tilde{E}\{(\rho_{t+1}-1)^2 \mid \Im_t\} < \infty.$$

By lemma $A.5$, the true measure is absolutely continuous with respect to the belief. Therefore, the belief and the true measure merge. *q.e.d.*

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