GAMES AND ECONOMIC BEHAVIOR **18,** 116–134 (1997)
ARTICLE NO. GA970522

Repeated Large Games with Incomplete Information*

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Received October 13, 1994

We examine repeated games with incomplete information where the type spaces of the players may be large. It is shown that if the belief of each player, regarding future play of the game, accommodates the true play then a Nash equilibrium of the incomplete information game will evolve, with time, into an equilibrium of the complete information game, i.e., the realized game where the types of all players are common knowledge. We introduce the notion of accommodating beliefs which involves two requirements. The first is that the belief assigns positive probability to neighborhoods of the true distribution and the second is that what lies outside of a neighborhood is separated from the true distribution by sufficient incoming observations (this is the separation property defined in the paper).

1. INTRODUCTION

Consider a repeated game with incomplete information. Each one of n players is selected, according to a known distribution, from a pool of

* We thank Alvaro Sandroni for valuable remarks as to the content, the proofs, and the presentation of this paper.

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0899-8256/97 $25.00

possible types. A player knows his own type but he does not know other players' true types. As the game evolves, all players' actions are observed and these partially reveal the true types that stand behind the strategies played.

Nash equilibrium of the incomplete information game is a strategy prescribed to each type of each player so that any player (no matter what type he is) is playing a best response to the opponents' expected strategy. Generally, the strategies actually played (those attached to the selected types) do not form a Nash equilibrium, because each individual strategy is not a best response to the particular strategies employed, but rather to the expected strategies. Consequently, the players' forecast regarding the play path of the game is generally wrong.

As time passes and the game is played repeatedly, each player's (realized) stage strategy is revealed. This enables all other players to update their forecast on the other players' strategies in the repeated game. Simultaneously, the players update their forecast on the distributions of the future play path of the game. In some situations the above described dynamics may enable the players to learn their opponents' strategies, at least on the play path. This phenomenon has been observed in games with incomplete information where the type sets are either finite or countable. See Aumann and Maschler (1966), Mertens and Zamir (1972), Sorin (1980), Hart (1985), and Kalai and Lehrer (1993a).

Kalai and Lehrer (1993a) introduced a model where players play optimally against their subjective beliefs regarding other players' strategies. In their model the strategies played must not necessarily form an equilibrium either of the game actually played or of any incomplete information game containing it. Kalai and Lehrer showed that, when the true strategies are absolutely continuous with respect to the beliefs, players learn to play Nash equilibrium. In particular, this implies that when the type space of each player is countable learning will occur.

In the case of a large type set, say, with a continuum of types, each type is typically assigned *a priori* zero probability. If different types have significantly different strategies, then the Kalai–Lehrer assumption (i.e., absolute continuity) is violated and their model does not apply. This, however, does not say that their results are wrong as we show in the following example.

Suppose one player is choosing between two actions, A and B, by flipping a $(p, 1-p)$ coin. Suppose that the type of this player is exactly his parameter p and moreover that p was chosen from a uniform distribution over the unit interval. As time goes by and all other players see this player's stage action, they get a good idea of the coin being used, although initially (and throughout the game as well) the probability assigned to that specific coin is zero. Learning does take place although the Kalai–Lehrer absolute continuity condition is violated. We shall return to this example again in the third section.

In specific examples of large type sets where the strategies of the different types collude asymptotically, the Kalai–Lehrer model and results do apply (for a specific example, see Smorodinsky (1995)).

In analyzing the case of large type sets, the techniques used by Kalai and Lehrer, i.e., the Martingale convergence theorem and the Radon–Nykodim theorem, do not yield the desired results. Therefore, one needs to introduce a new technique to tackle the problem. Indeed, one of the main tasks of this paper is to develop new mathematical tools that can be used for Bayesian learning in cases where the truth is assigned zero probability.

This paper adopts the Kalai–Lehrer (1993a, Section 6) model and deals with a large type space. We introduce a condition on the beliefs over the type space, which we call accommodation, and show that if the beliefs *accommodate* the truth then strategies of the incomplete information game converge to the true strategies played.

The intuition behind the notion of accommodation is quite similar to the intuition behind the Kalai–Lehrer notion of "grain of truth." Kalai and Lehrer show that if the true distribution is assigned positive weight by the belief then in the long run as information is accumulated the belief puts almost all weight on the truth. Similarly here we require that a neighborhood of the truth is assigned positive probability, even when the true distribution itself has zero probability. More precisely, in order to accommodate the true distribution, the belief must satisfy two conditions. First, it must assign any neighborhood of the true type (containing similar types—of those inducing close strategies) positive probability. And the second condition is that what lies outside of the neighborhood is separated from the truth. What we mean by this is that there will be sufficiently many times where the true distribution and the outside-of-neighborhood distribution induce significantly different predictions over the near future outcomes. This property enables an observer to separate the distributions that are not in the vicinity of the truth from the truth itself. We therefore call it the *separation* property.

After presenting the formal model of repeated games with incomplete information (in Section 2), we discuss a general setup of beliefs updating (in Section 3). Two distributions, interpreted as the belief and the truth, are updated as more realizations are observed. We say that the belief weakly merges (see Kalai and Lehrer (1994)) to the true distribution if with probability one (with respect to the true distribution) the predictions over short-run events of the updated belief and of the updated true distribution become close as time goes by. We show that if the belief accommodates the true distribution, then the belief weakly merges to the true distribution. This result extends the well-known merging of opinions theorem of Blackwell and Dubins (1962).

Since the technique we utilize is new in the literature of Bayesian updating we devote a special section (Section 4) to the proofs. Instead of the Martingale convergence theorem we use the strong law of large numbers, which is conceptually simpler than the first, applied to uncorrelated random variables.

Application of the general theory of beliefs updating to games with incomplete information is another central issue of the paper. In Section 5 we show that if the type space is accommodating, then the expected updated strategies (subject to the uncertainty resulted from the incomplete information) converge to Nash equilibrium of the true game (corresponding to the particular types selected).

Rational learning in discounted repeated games with incomplete information has also been treated by Nyarko (1994, 1996), Koutsougeras and Yannelis (1994), and Jordan (1995). In these papers there are two distinct approaches; the first studies convergence results of the players' beliefs and the second, similar to what we discuss here, is about convergence of the actual strategies played. The similarity between the latter two approaches is that they are confined to within-equilibrium context; all these papers, including ours, investigate equilibrium of the incomplete information game and show convergence to equilibrium of the realized game.

The issue of relaxing the absolute continuity hypothesis is also central to Sandroni (1996). In his work, Sandroni takes a different approach to show that one can obtain convergence of strategies to Nash equilibrium under a weaker condition defined as "almost absolute continuity." Whereas the motivation for our work has been the search for weaker compatibility conditions other than those used by Blackwell and Dubins, Sandroni is interested in taking a trembling approach (Sandroni considers trembles on players' beliefs) to show that (almost) absolute continuity is actually weaker than one might think and is actually weak enough to be equivalent to convergence to Nash equilibrium in Sandroni's setup.

We close the paper with a short final comment which points out some possible generalizations and applications of our results.

2. THE MODEL

2a. *An Infinitely Repeated Game*

We start by describing a model of an n-person infinitely repeated game with incomplete information. We use the following notation.

T_i	Player i's type space ($i = 1, \dots, n$). T_i may be finite, countable, or even a continuum. (The novelty of the paper is in introducing the possibility for large type space.)
ν_i	A probability measure on T_i (given a proper σ-field), according to which the specific type of player i is chosen ($i = 1, \dots, n$). We assume throughout that ν_i and ν_j are independent for $i \neq j$. Naturally, $\prod_{i=1}^n \nu_i$ is a probability measure on $\prod_{i=1}^n T_i$. We denote this measure by ν.
A_i	The action space for player i at each stage of the game. We allow for A_i to be either finite or countable.
$A = \prod_{i=1}^n A_i$	The product action space.
A^l	The set of histories of length $l (l = 1, 2, \dots)$.
$\hat{A} = \bigcup_{l=1}^\infty A^l$	The set of all finite histories.
$A^\infty = \prod_{s=1}^\infty A$	The set of all infinite play paths.
$u_i^{t_i}: A \to \mathbb{R}$	The utility function of type t_i ($t_i \in T_i$) of player i, at each stage of the game.
λ_i	The discount factor of player $i, 0 < \lambda_i < 1$ for all $i = 1, \dots, n$.[1]

2b. *Strategies of the Repeated Game*

Let $\hat{f}_i: \hat{A} \to \Delta(A_i)$ be a behavior strategy of player i. Namely, player i, after observing any finite history, chooses a lottery on A_i in order to determine his next action. Denote by F_i the set of all behavior strategies. By Kuhn's theorem (1953) this is equivalent to the set of all mixed strategies. Given a specific finite history, define a cylinder set in F_i as a set of strategies that prescribe the same lottery given that history. We always consider the σ-algebra on F_i generated by all cylinder sets.

A strategy plan for player i is a function prescribing to every type t_i in T_i a behavior strategy, $f_i: T_i \to F_i$. f_i is restricted to be a measurable function. We will denote by $f_i^{t_i}$ the specific strategy assigned to type t, namely, $f_i^{t_i} \equiv f_i(t_i)$. Denote by f the joint strategy (or strategy profile) $(f_1, \dots, f_n)$ and by f_{-i} the same profile without i's strategy. The measure ν and a joint strategy plan f induce a distribution over A^∞ which we call μ_f (as ν is fixed throughout, we omit it from the notation).

The type profile, $t = (t_1, \dots, t_n)$, of all players is chosen at random using the measure ν. Its ith component, t_i, player i's type, becomes private

[1] The results of the paper can be extended to the case where the discount factor is type contingent.

information of player i. From player i's point of view, after his type t_i has been realized, μ_f is no longer the probability measure on the true play path. It is rather the conditional of μ_f given t_i. We shall denote this by $\mu_{f \mid t_i}$, and call it the belief of type t_i of player i.

Denote by a^s the joint action at time s. When f is played the expected, long-run, payoff for player i, after realizing his own type t_i is

$$U_i^{t_i}(f) = E_{\mu_{f \mid t_i}}\left(\sum_{s=1}^{\infty} \lambda_i^s u_i^{t_i}(a^s) \right).$$

We go on to define a Nash equilibrium in the incomplete information game.

DEFINITION 2.1. f is a *Nash equilibrium* if $\forall\, i = 1, \dots, n$, $\forall\, t_i \in T_i$, $f_i^{t_i}$ is a best response against f_{-i}, where f_{-i} is the vector of strategies assigned to all players but i.

Another important notion is the complete information Nash equilibrium, i.e., an equilibrium of the game where all the selected types $t_i \in T_i$ are common knowledge. Let $t = (t_1, \dots, t_n)$ be any n-tuple of types and let $f^t = (f_1^{t_1}, \dots, f_n^{t_n})$ denote the corresponding joint strategy. This joint strategy also induces a measure over A^∞, denoted by μ_{f^t}. We call μ_{f^t} the true measure when t is realized. The complete information expected payoff of player i is

$$U_i(f^{t_i}) = E_{\mu_{f^t}}\left(\sum_{s=1}^{\infty} \lambda_i^s u_i^{t_i}(a^s) \right).$$

DEFINITION 2.2. The joint strategy f^t is a *complete information Nash equilibrium* (CINE) if $f_i^{t_i}$ is a best response to all other players' true strategy, i.e., to $(f_j^{t_j})_{j \neq i}$.

Our aim is to show that under some plausible conditions a Nash equilibrium (of the incomplete information game) will converge in time to a complete information Nash equilibrium. In order to show such convergence, the notion of induced strategy and some notion of a distance between joint strategies must be introduced.

2c. *The Induced Strategy*

Suppose that $f_i^{t_i}$ is the strategy prescribed to type t_i of player i. After any finite history $h \in A^l$ the induced strategy $f_i^{t_i}(h)$ is defined to be the continuation of $f_i^{t_i}$ after h. Note that $f_i^{t_i}(h)$ is itself a strategy for player i in the infinitely repeated game.

The updated belief (using Bayes' rule) of type t_i of player i after history h is denoted by $\mu_{f|t_i}(\cdot \mid h)$. The updated true measure is $\mu_{f^t}(\cdot \mid h) = \mu_{f^t(h)}$.

In the game described, the players start out by optimizing given their belief on the future (each player has his own belief given his realized type). We examine whether the players learn the true measure with time and consequently optimize against each other's true type, leading the game into a CINE.

2d. *The Distance between Two Joint Strategies*

We adopt here the approach of Kalai and Lehrer (1993a,b) in measuring the distance between two given joint strategies.

DEFINITION 2.3. Two joint strategies f and g are said to *play* (ε, l)*-like* each other, for given $\varepsilon > 0$ and $l \in \mathbb{N}$, if $|\mu_f(B) - \mu_g(B)| < \varepsilon$, $\forall B \subset A^l$.

Thus, f and g play (ε, l)-like each other if the probabilities assigned by both to any event consisting of strings of l actions are ε-close.

3. WEAK MERGING OF MEASURES

In the previous section we defined, for a given joint strategy plan f and an n-tuple of type realizations, $n + 2$ measures on the set of all play paths. μ_f denoted the measure induced by f; μ_{f^t} denoted the true measure, given the realized types profile $t = (t_1, \dots, t_n)$ (note that μ_{f^t} is known to no player); $\mu_{f|t_i}$ denoted the belief of type t_i of player $i, i = 1, \dots, n$.

We now analyze the connection between a player's belief on the future outcome of the repeated game, and the true distribution. In order to make the analysis more general we look at a general measurable space $(\Omega, \mathscr{F})$, two measures on $(\Omega, \mathscr{F})$ denoted by μ and $\tilde{\mu}$, referred to as the true measure and the belief, respectively, and a filtration[2] $\{\mathscr{P}_s\}_{s=1}^{\infty}$. Throughout this section the filtration is fixed and all that is stated is relative to this specific filtration. Typically, what may be true for one filtration may fail to be true for another. In later use $\mathscr{P}_s$ will be the partition over the set of play paths generated by histories of length s. Thus, an agent in a game with perfect monitoring is informed of the history, namely, the atom of $\mathscr{P}_s$ where the realized play path is located.

DEFINITION 3.1. Given $\omega \in \Omega$ we say that $\tilde{\mu}$ is ε*-close* to μ at stage s if $\tilde{\mu}(\cdot \mid P_{s-1}(\omega))$ and $\mu(\cdot \mid P_{s-1}(\omega))$ are $(\varepsilon, 1)$ close to each other, where $P_{s-1}(\omega)$ is the atom of $\mathscr{P}_{s-1}$ containing ω.

[2] A *filtration* $\{\mathscr{P}_s\}_{s=1}^{\infty}$ is a sequence of increasing ($\mathscr{P}_s$ refines $\mathscr{P}_{s-1}$) finite or countable partitions of Ω that generate $\mathscr{F}$.

DEFINITION 3.2. $\tilde{\mu}$ *weakly merges* to μ (denoted $\tilde{\mu} \overset{\text{WM}}{\to} \mu$) if for any $\varepsilon > 0$, and for μ-almost every ω there is a time $S = S(\varepsilon, \omega)$ such that $s \geq S$ implies that $\tilde{\mu}$ is ε-close to μ at stage s.

In other words, $\tilde{\mu}$ weakly merges to μ if for any $\varepsilon > 0$ the one-step-ahead forecasts of $\tilde{\mu}$ and of μ are close up to an ε from some time on.

Remark 3.1. Notice that weak merging implies that the forecast of μ and $\tilde{\mu}$ will eventually be close for any finite horizon. In other words, Definition 3.1 and, therefore, Definition 3.2 could require (ε, l) closeness, where l is any fixed integer, but $l = 1$ is sufficient.

One way in which a belief $\tilde{\mu}$ is generated is where μ is chosen randomly from some set of possible distributions, and the agent is simply uninformed of which measure was actually chosen. This can be formalized in a parametric model. For this reason we shall work simultaneously with the beliefs $\tilde{\mu}$ in the parametric form and with the general case. To formalize $\tilde{\mu}$ in the parametric form consider the measurable space of parameters $(\Theta, \mathscr{H})$ with a given distribution F. The true measure is chosen by selecting a parameter $\theta_0 \in \Theta$ according to the distribution F and taking $\mu = \mu_{\theta_0}$. For any measurable set $\hat{\Theta} \in \mathscr{H}$ with positive probability we use the notation

$$\mu_{\hat{\Theta}} \equiv \frac{1}{F(\hat{\Theta})} \int_{\hat{\Theta}} \mu_\theta \, dF(\theta)$$

(note that using this notation $\tilde{\mu} = \mu_\Theta$). Throughout this section we fix the σ-algebra $\mathscr{H}$ and the distribution F on Θ, and all definitions regarding the parameter space Θ are with respect to $\mathscr{H}$ and F.

Note that, in our model of a game with incomplete information, the parametric version is the natural way to model beliefs.

We now proceed to use the notions of asymptotic ε-nearness and separation to state sufficient conditions for weak merging.

DEFINITION 3.3. We say that the distribution $\tilde{\mu}$ is *asymptotically ε-near* μ if with μ-probability 1 there is a random time S s.t. $s \geq S$ implies

$$\left| \frac{\tilde{\mu}(P_{s+1} \mid P_s)}{\mu(P_{s+1} \mid P_s)} - 1 \right| < \varepsilon$$

(with $0/0$ defined to be 1).

DEFINITION 3.4a. A measure μ' has the *separation property* with respect to μ if there exists $d > 0$ such that the set

$$\left\{s \in \mathbb{N} \mid \exists A \in \mathscr{P}_{s+1} \text{ s.t.} \left|\mu'\left(A \mid P_s(\omega)\right) - \mu\left(A \mid P_s(\omega)\right)\right| > d\right\}$$

has positive lower density, μ-a.e.[3]

In words, μ' has the separation property w.r.t. μ if with μ-probability 1 there will be many stages s (with positive lower density) such that the updated forecasts of μ' and of μ are bounded away from each other (by at least d). That is, there is at least one short-run event, $A \in \mathscr{P}_{s+1}$, regarding which μ' and μ significantly differ (i.e., $|\mu'(A \mid P_s(\omega)) - \mu(A \mid P_s(\omega))| > d$).

DEFINITION 3.4b (The Parametric Version). A subset $\hat{\Theta} \subset \Theta$ has the *separation property* w.r.t. $\theta_0 \in \Theta$ if $\mu_{\hat{\Theta}}$ has the separation property w.r.t. μ_θ.

DEFINITION 3.5a. A belief $\tilde{\mu}$ is said to *accommodate* μ if, for any $\varepsilon > 0$, $\tilde{\mu}$ can be decomposed as follows: $\tilde{\mu} = \alpha\mu_\varepsilon + (1 - \alpha)\nu_\varepsilon$ for some $\alpha \in (0, 1]$, where μ_ε, ν_ε are probability measures satisfying

(i) μ_ε is asymptotically ε-near μ.

(ii) ν_ε is a convex combination of ν_ε^j $(1 \le j \le M)$, $\nu_\varepsilon = \sum \beta_j \nu_\varepsilon^j$, where ν_ε^j is a probability measure which has the separation property w.r.t. μ.

The meaning of part (i) is that the belief $\tilde{\mu}$ is diffused around the true μ. It need not assign positive probability to μ, yet it does assign positive probability to any neighborhood μ_ε of μ. Part (ii) requires that on a significant (i.e., with positive lower density) set of stages s the predictions of ν_ε^j and of μ are different. To clarify the significance of part (ii) we give the following example (from Kalai and Lehrer (1994) and Lehrer and Smorodinsky (1996)) in which the belief $\tilde{\mu}$ satisfies the first part of the definition, but does not satisfy the second part. In this example weak merging is not achieved.

EXAMPLE 3.1. Let Ω be the space $\{0, 1\}^{\mathbb{N}}$, and let $\mathscr{P}_s$ be the partition induced by the first s coordinates. Define μ to be the Dirac measure on the point $(1, 1, \dots)$. Define $\tilde{\mu}$ to be the measure $(1/2)\mu_1 + (1/2)\mu_2$, where μ_1 and μ_2 are defined as follows. μ_1 is the measure induced by the sequence $X_1, X_2, \dots$ of independent Bernoulli random variable, where $\text{prob}(X_s = 1)$ is 1 if $s \neq 2^{2^k}$, and it is $1/2$ if $s = 2^{2^k}$. The measure μ_2 is the one induced as follows. Denote by μ_2^m the measure induced by the i.i.d.

[3] The *lower density* of $L \subseteq \mathbb{N}$ is defined by $\liminf_{m \to \infty} \#\{L \cap \{1, \dots, m\}\}/m$.

process assigning probability $(1 - 1/m)$ to 1 at every period. Let $\mu_2 = \sum_{m=1}^{\infty} (1/2^m)\mu_2^m$. In other words, with probability $(1/2^m)$ $(m = 1, 2, \ldots)$ it is defined by a repeated toss of a coin assigning probability $(1 - 1/m)$ to 1.

One can show that, after observing $s = 2^{2^k} - 1$ times the outcome 1, the updated measure of $\tilde{\mu}$ assigns most of the weight to μ_1 and therefore assigns a probability close to 1/2 to the event that the next outcome will be 1 while the updated measure of μ assigns the same event probability 1. Thus, $\tilde{\mu}$ does not weakly merge to μ. In this example part (i) of Definition 3.5a is satisfied (to see this let μ_ε be the convex combination of all coins assigning probability larger than $(1 - \varepsilon)$ to the event 1) whereas part (ii) is violated. This is because μ_1 assigns different probabilities from those of μ on a set having density zero. So μ_1 is not separated from μ.

DEFINITION 3.5b (The Parametric Version). Θ is an *accommodating set of parameters* if $\forall \theta_\theta \in \Theta$, $\forall \varepsilon > 0$, $\exists N(\theta_0, \varepsilon) \subset \Theta$, which we shall call a neighborhood of θ_0, satisfying:

(i) $\forall \theta \in N(\theta_0, \varepsilon)$ μ_θ is asymptotically ε-near μ_{θ_0};

(ii) $F(N(\theta_0, \varepsilon)) > 0$;

(iii) $N(\theta_0, \varepsilon)^c$ can be finitely partitioned as follows: $N(\theta_0, \varepsilon)^c = \bigcup_{j=1}^{M} N_j$ s.t. $\forall j = 1, \ldots, M$, N_j has the separation property w.r.t. θ_0, and $F(N_j) > 0$.

We will claim that accommodation is a sufficient condition for weak merging.

Remark 3.2. In a case where μ is absolutely continuous[4] with respect to $\tilde{\mu}$, $\tilde{\mu}$ has the separation property w.r.t. μ. Simply take, for every $\varepsilon > 0$, $\mu_\varepsilon = \tilde{\mu}$ and $\alpha = 1$. It can be shown, by an application of the Martingale convergence theorem (see Shiryayev (1984)), that the quotient $\tilde{\mu}(P_{s+1}(\omega) \mid P_s(\omega))/\mu(P_{s+1}(\omega) \mid P_s(\omega))$ converges to 1 (for a complete proof, the reader is referred to Proposition 2 in Kalai and Lehrer (1993a)). Therefore, $\tilde{\mu}$ is asymptotically ε-near μ for any $\varepsilon > 0$.

EXAMPLE 3.2. The following example captures the main idea behind the notions presented. Suppose that a coin with parameter α is repeatedly and independently tossed to generate an infinite sequence of Heads and Tails. Let μ_α describe the distribution induced by this coin. An agent, not knowing the true parameter of the coin, has a belief $\tilde{\mu}$, according to which α was chosen. Suppose that $\tilde{\mu}$ is the uniform distribution over the interval of possible parameters (0, 1). It is readily seen that the true distribution over infinite strings of Heads or Tails is not absolutely continuous with

[4] μ is *absolutely continuous* with respect to $\tilde{\mu}$ if for every event A, $\mu(A) > 0$ implies $\tilde{\mu}(A) > 0$.

respect to the belief. For example, $\tilde{\mu}$ assigns probability zero to the event that the asymptotic frequency of Heads is precisely α, while the true probability for this event is 1.

However, as tosses of the coin are observed, the posteriors over the possible parameters become concentrated around α. The point α, though, always attains zero probability and, therefore, the lack of absolute continuity remains in effect all the time.

Despite the lack of absolute continuity, after a sufficiently long time one can predict with high precision the distribution over the subsequent set of possible outcomes; it is approximately α over Heads and Tails. Bayesian updating enables the observer to learn the correct distribution over short-run outcomes. But there is no way to learn to predict infinite horizon events, as in the case of absolute continuity. This, however, should represent no drawback for agents who discount future utility.

What makes the learning possible in the coin example is that the parameter set $\Theta = (0, 1)$ is accommodating. Let $(\alpha - \varepsilon/2, \alpha + \varepsilon/2)$ be an ε-neighborhood of the true α. Notice that for any α' in this neighborhood $\mu_{\alpha'}$ is $\varepsilon/2$-asymptotically close to μ. Moreover, this neighborhood is assigned probability $\varepsilon > 0$ according to the prior distribution on α and the complement of this neighborhood is $N_1 \cup N_2 = (0, \alpha - \varepsilon/2\,] \cup [\,\alpha + \varepsilon/2, 1)$, where both N_1 and N_2 have the separation property w.r.t. α. For any parameter α' on the left side of α, i.e., $\alpha' \in (0, \alpha - \varepsilon/2]$, the probability w.r.t. $\mu_{\alpha'}$ of Heads is smaller than the one assigned by μ_α by more than $\varepsilon/2$. Therefore, the set $N_1 = (0, \alpha - \varepsilon/2]$ has the separation property. For the same reason, $N_2 = [\alpha + \varepsilon/2, 1)$ has the separation property. Therefore, the complement of the ε-neighborhood of α satisfies (iii) of Definition 3.5b.

THEOREM A. *If $\tilde{\mu}$ accommodates μ then $\tilde{\mu} \overset{\text{WM}}{\rightarrow} \mu$.*

The parametric version of Theorem A is:

THEOREM B. *If Θ is an accommodating set of parameters then $\forall \theta \in \Theta$ $\mu_\Theta \overset{\text{WM}}{\rightarrow} \mu_\theta$.*

4. PROOFS OF THEOREMS A AND B

We now turn to prove Theorems A and B. The proof of Theorem A is essentially an application of the strong law of large numbers and the following two lemmata:

LEMMA 1. *Let $\{p_i\}$ and $\{q_i\}$ be two sequences of nonnegative numbers that sum to at most 1. Then, $\Sigma_i |p_i - q_i| \geq d$ implies $\Sigma_i \sqrt{p_i q_i} \leq 1 - d^2/8$.*

Proof. See Appendix.

LEMMA 2. *Suppose* μ_ε *is asymptotically* ε*-near* μ, *then* μ a.e. *there is a random time* S, s.t. $s \geq S$ *implies* μ_ε *is* ε*-close to* μ *at stage* s.

Proof. See Appendix.

Proof of Theorem A. Fix an $\varepsilon > 0$ and let μ_ε and ν_ε be as in Definition 3.5a. Fix an i between 1 and M and define

$$X_s(\omega) = \left[\frac{\nu_\varepsilon^i(P_s(\omega) \mid P_{s-1}(\omega))}{\mu(P_s(\omega) \mid P_{s-1}(\omega))}\right]^{1/2}$$

and $Y_s = X_s - E(X_s \mid \mathscr{P}_{s-1})$. $\{Y_s\}_{s=1}^{\infty}$ is a sequence of uncorrelated random variables with a finite second moment. Therefore, the strong law of large numbers may be applied (see Chung (1974), p. 103) and the conclusion is that there exists a sequence δ_s diminishing to zero such that

$$(1/S)\sum_{s=1}^{S} X_s \leq (1/S)\sum_{s=1}^{S} E(X_s \mid \mathscr{P}_{s-1}) + \delta_S, \qquad \forall S. \tag{1}$$

By a straightforward application of the inequality of averages:

$$\begin{aligned} &E(X_s(\omega) \mid P_{s-1}(\omega)) \\ &\quad = \sum_{P_s(\omega)} \left[\mu(P_s(\omega) \mid P_{s-1}(\omega)) \cdot \nu_\varepsilon^i(P_s(\omega) \mid P_{s-1}(\omega))\right]^{1/2} \\ &\quad \leq \sum_{P_s(\omega)} \frac{\mu(P_s(\omega) \mid P_{s-1}(\omega)) + \nu_\varepsilon^i(P_s(\omega) \mid P_{s-1}(\omega))}{2} = 1. \end{aligned} \tag{2}$$

Now, for a period s such that there is an $A \in \mathscr{P}_s(\omega)$ satisfying

$$|\nu_\varepsilon^i(A \mid P_{s-1}(\omega)) - \mu(A \mid P_{s-1}(\omega))| > d$$

(where d originates from the separation property of ν_ε^i and μ) we can show a stronger form of inequality. Denote by $\mathbb{N}^i \subseteq \mathbb{N}$ the set of all periods where this happens. By the separation property the lower density of $\mathbb{N}^i$, denoted $\underline{\text{dens}}(\mathbb{N}^i)$, is strictly positive, namely,

$$\exists\, \eta > 0 \text{ s.t. } \underline{\text{dens}}(\mathbb{N}^i) \geq \eta > 0. \tag{3}$$

Denote by p_j the probability of atoms of $\mathscr{P}_s(\omega)$ according to $\nu_\varepsilon^i(\cdot \mid P_{s-1}(\omega))$, and by q_j the probability of these atoms according to $\mu(\cdot \mid P_{s-1}(\omega))$. By the separation property $\Sigma|p_j - q_j| \geq |\nu_\varepsilon^i(A \mid P_{s-1}(\omega)) - \mu(A \mid P_{s-1}(\omega))| > d$. Therefore, by Lemma 1,

$$E(X_s(\omega) \mid P_{s-1}(\omega)) = \Sigma q_j \sqrt{\frac{p_j}{q_j}} = \Sigma(q_j p_j)^{1/2} \leq 1 - \frac{d^2}{8}. \tag{4}$$

As $\underline{\text{dens}}(\mathbb{N}^i) > \eta$ there exists a time S_0 s.t. $S \geq S_0$ implies

$$\frac{\#\left[\mathbb{N}^i \cap \{1,2,\dots,S\}\right]}{S} > \frac{\eta}{2} \qquad \text{and}$$

$$\frac{\#\left[(\mathbb{N} \setminus \mathbb{N}^i) \cap \{1,2,\dots,S\}\right]}{S} \leq 1 - \frac{\eta}{2}.$$

Combine this with (2), (3), and (4) to conclude that for $S \geq S_0$,

$$\frac{1}{S}\sum_{s=1}^{S} E\big(X_s(\omega) \mid P_{s-1}(\omega)\big) \leq \left(1 - \frac{\eta}{2}\right) + \frac{\eta}{2}\left(1 - \frac{d^2}{8}\right) < 1.$$

So there exists some $\beta > 0$ s.t.

$$\frac{1}{S}\sum_{s=1}^{S} E\big(X_s(\omega) \mid P_{s-1}(\omega)\big) \leq 1 - 2\beta, \qquad \forall S > S_0. \tag{5}$$

By (1) and (5), $(1/S)\sum_{s=1}^{S} X_S \leq 1 - 2\beta + \delta_s$. By taking S large enough, such that δ_S is smaller than β, we get that $(1/S)\sum_{s=1}^{S} X_s \leq 1 - \beta$. Hence, $\prod_{s=1}^{S} X_s \leq (1-\beta)^S$ (inequality of averages). But as $\prod_{s=1}^{S} X_s = (\nu_\varepsilon^i(P_s(\omega))/\mu(P_s(\omega)))^{1/2}$, we conclude that $\nu_\varepsilon^i(P_s(\omega))/\mu(P_s(\omega)) \leq (1-\beta)^{2s}$, for some $\beta > 0$, and for $S > S_0$.

As was shown in Lehrer and Smorodinsky (1996) (see proof of Corollary 1 there), property (i) in the definition of accommodation (Definition 3.5a) guarantees that

$$\left(\frac{\tilde{\mu}(P_s(\omega))}{\mu(P_s(\omega))}\right)^{1/S} \to 1 \;\; \mu\text{-a.s.} \tag{6}$$

(For the sake of completeness we state and prove this claim as Lemma 3 in the Appendix.) We now obtain

$$\begin{aligned} \frac{\nu_\varepsilon^i(P_s(\omega))}{\tilde{\mu}(P_s(\omega))} &= \frac{\nu_\varepsilon^i(P_s(\omega))}{\mu(P_s(\omega))}\,\frac{\mu(P_s(\omega))}{\tilde{\mu}(P_s(\omega))} \\ &\leq (1-\beta)^{2s}\left[\frac{\mu(P_s(\omega))}{\tilde{\mu}(P_s(\omega))}\right] \\ &= \left[(1-\beta)^2\left(\frac{\mu(P_s(\omega))}{\tilde{\mu}(P_s(\omega))}\right)^{1/s}\right]^s \sim (1-\beta)^{2s} \to 0 \;\; \mu\text{-a.s.} \end{aligned} \tag{7}$$

As (7) holds for any $i = 1, \ldots, M$ and since ν_ε is a finite convex combination of ν_ε^i, we obtain that $\nu_\varepsilon(P_s(\omega))/\tilde{\mu}(P_s(\omega))$ also converges to 0 μ-a.s. Since $\tilde{\mu} = \alpha\mu_\varepsilon + (1-\alpha)\nu_\varepsilon$ we conclude that $\forall\varepsilon > 0$ $\mu_\varepsilon(P_s(\omega))/\tilde{\mu}(P_s(\omega)) \to 1/\alpha$ μ-a.e. This in turn implies $\tilde{\mu}(P_{s+1}(\omega) \mid P_s(\omega))/\mu_\varepsilon(P_{s+1}(\omega) \mid P_s(\omega)) \to 1$ μ-a.e. Combining this with the asymptotic ε-nearness property of μ_ε it is concluded that $\tilde{\mu}$ is asymptotically ε-near μ for any $\varepsilon > 0$. By Lemma 2, $\tilde{\mu}$ is eventually ε-close to μ, μ-a.s. $\forall\varepsilon > 0$, which implies $\tilde{\mu} \overset{\text{WM}}{\to} \mu$. ■

We now turn to the proof of Theorem B using the same technique as in the proof of Theorem A.

Proof of Theorem B. Take $\theta_0 \in \Theta$ and define $\mu_\varepsilon = \mu_{N(\theta,\varepsilon)}$ and $\nu_j = \mu_{N_j}$, where $\bigcup_{j=1}^{M} N_j = N(\theta,\varepsilon)^c$ (as in part (iii) of Definition 3.5b). It is straightforward to see that μ_ε satisfies condition (i) of Definition 3.5a. Now repeat the technique of the previous proof to show that $\forall j = 1, \ldots, M$, $\mu_{N_j}(P_s(\omega))/\mu_\Theta(P_s(\omega)) \to 0$ μ_{θ_0}-a.s. With this in hand it can be concluded (again, by repeating the final argument in the Proof of Theorem A) that μ_Θ is asymptotically ε-near μ_{θ_0} for any $\varepsilon > 0$, which, by using Lemma 2, gives the desired result, namely, $\mu_\Theta \overset{\text{WM}}{\to} \mu_{\theta_0}$. ■

5. LARGE GAMES WITH INCOMPLETE INFORMATION

Let T_i, ν_i, A_i, u_i^t, λ_i, $i = 1, \ldots, n$ be the components of the one-shot incomplete information game, as described in Section 2. Let $f = (f_1, f_2, \ldots, f_n)$ be a Nash equilibrium of the incomplete information repeated game. Fix a vector of realized types $t = (t_1, \ldots, t_n)$.

We denote by μ_{f^t} the probability measure on A^∞ induced by f^t, the realized strategy, and $\mu_{f \mid t_i}$ by the belief held by type t_i of player i. For any play path $\omega \in A^\infty$ and for every period s we denote by ω^s the s first coordinates of ω (i.e., its s-prefix).

The main contribution of this paper asserts that once the game evolves according to a Bayesian equilibrium and the accommodation condition is satisfied, then on the play path the strategies converge to strategies of the complete information Nash equilibrium. Note that the convergence is with respect to the metric defined in Definition 2.3. Formally, the result is:

THEOREM C. *If f is a Nash equilibrium of the incomplete information game, and if any $t \in \prod_{i=1}^n T_i$ $\prod_{i=1}^n T_i$ is an accommodating set of parameters then for every $\varepsilon > 0$ and any $l > 0$ there is a (random) time S* s.t. *if $s > S$ then μ_{f^t}*-a.e. *ωA^∞, $f^t(\omega^s)$ plays (l, ε)-like a* CINE.

The main idea behind the proof is to use Theorem B and to show that the belief each player holds, eventually, regarding the future play of the game is quite similar, when looking at near future events, to the true distribution. We combine this with the compactness of the strategy space to get the desired result. In order to formalize the proof of Theorem C, we need one more definition and a lemma. Following Kalai and Lehrer (1993b), we define the following.

DEFINITION 5.1. A strategy profile $\{f^{t_i}\}_{i=1}^{n}$ $(t_i \in T_i)$ is said to generate a *subjective* (ε, l)*-equilibrium* if $f_i^{t_i}$ is a best response to f_{-i} and, furthermore, $(f_1, f_2, \ldots, f_i^{t_i}, \ldots, f_n)$ plays (ε, l)-like $(f_1^{t_1}, f_2^{t_2}, \ldots, f_n^{t_i})$. If $\{f^{t_i}\}_{i=1}^{n}$ generates a subjective (ε, l)-like equilibrium for all $\varepsilon > 0$ and $l \in \mathbb{N}$, then it is a *subjective equilibrium.*

A strategy profile is a function from all finite histories, $\hat{A}$, to mixed-action profiles. Thus, since there are countable many histories, a strategy profile can be presented as a sequence of one-shot mixed action profiles: one for each history. The set of these sequences, $(\times_{i=1}^{n} \Delta(A_i))^{\infty}$, is compact in the product topology. Thus, any sequence of strategy profiles $\{f_k\}_{k=1}^{\infty}$ has a converging subsequence $\{f_{k_l}\}_{l=1}^{\infty}$. For any history $h \in \hat{A}$, $f_{k_l}(h)$ converges to, say, $f(h)$. This, in turn, gives us that any sequence of n-tuples of strategy profiles (a strategy profile for each player, representing his own strategy and his beliefs on the others' strategies) has a converging subsequence. Specifically, any sequence $\{f_k\}_{k=1}^{\infty}$ of subjective (η_k, n_k)-equilibria will converge (taking a subsequence, if necessary) to an n-tuple of strategy profiles, f. Moreover, if the η_k converges to 0 and n_k to infinity, then f must itself be a subjective equilibrium. Finally, any subjective equilibrium plays $(0, l)$-like some CINE (see Kalai and Lehrer (1993b, Proposition 1)), for any integer l. This observation will be helpful in showing the following:

LEMMA 5.1. *For any* $\varepsilon > 0$ *and* $l \in \mathbb{N}$ *there exist* $\bar{\eta} > 0$, $\bar{n} \in \mathbb{N}$ s.t. *for all* $\eta < \bar{\eta}$ *and* $n > \bar{n}$ *every subjective* (η, n)*-equilibrium plays* (ε, l)*-like some* CINE.

Proof. Assume the conclusion of the lemma is wrong. Thus, there exists a sequence $\{f_k\}$ of subjective (η_k, n_k)-equilibria, for some $\eta_k \to 0$ and $n_k \to \infty$, where each f_k does not play (ε, l)-like any CINE. The limit f of a converging subsequence plays $(0, l)$-like some CINE.

When k is large enough, f_k plays (ε, l)-like f, which plays like a CINE. Therefore, f_k plays (ε, l)-like a CINE, thus contradicting the assumption. ∎

We now turn to the proof of Theorem C.

Proof of Theorem C. This will be based on Theorem B and Lemma 5.1. Fix (ε, l) and use Lemma 5.1 to compute (η, n) for which any subjective (η, n)-equilibrium will play (ε, l)-like some CINE. Now let t_i be the true type of player i. Player i's belief on the strategies of his opponents is f_{-i} and the measure induced on the space of all play paths is $\mu_{f \mid t_i}$. The game truly evolves according to $f^t = (f_1^{t_1}, f_2^{t_2}, \dots, f_n^{t_n})$ with the measure μ_{f^t}. As $\prod_{i=1}^n T_i$ is an accommodating set we know by Theorem B that $\mu_{f \mid t_i}$ weakly merges to μ_{f^t}. Since we begin with a Bayesian equilibrium, we also have that $f_i^{t_i}$ is a best response to f_{-i}. We therefore can conclude that there exists S_0 such that $S > S_0$ implies that $(f_{-i}, f_i^{t_i})(\omega^s)$ plays (η, n)-like $f^t(\omega^s)$ for all $i = 1, \dots, n$ and therefore $(f_i^{t_i}(\omega^s))_{i=1}^n$ is a subjective (η, n)-equilibrium. By the way (η, n) were computed, we conclude that $(f_i^{t_i}(\omega^s))_{i=1}^n$ plays (ε, l)-like a CINE for all $S > S_0$. ■

6. CONCLUSION

We have shown that initial beliefs in an incomplete information game may imply that players learn the expected behavior of each other and therefore learn to play optimally against the true selection of players. To this end we introduced the notion of accommodating belief which involves diffusion over neighborhoods of the true distribution and separation from the neighborhood complement.

6.1. *Compatibility of Accommodation and Optimization*

Nachbar (1995) provides a class of 2×2 games and sets of strategies where the notion of learning, which is a consequence of accommodation, is in contrast to mutual rationality, i.e., best response. Therefore, an interesting direction to pursue is to find natural circumstances in which one may expect accommodating beliefs and mutual rationality (best response) at the same time. One may ask, for instance, When does the set of rationalizable strategies comply with accommodation?

Another interesting issue is the case when players are restricted to a small set of strategies (e.g., bounded recall strategies). How small can such a set be while still satisfying the accommodation assumption?

6.2. *"Out-of-Equilibrium" Discussion*

Kalai and Lehrer (1993a) and Nyarko (1996) do not confine the discussion to "within equilibrium" but rather begin with strategies that are not in equilibrium. Nyarko (1996) focuses on a subjective Bayesian equilibrium without common priors. Instead of having a common prior, Nyarko relaxes his assumption, as in Kalai and Lehrer (1993a), to absolute continuity between beliefs and true strategies. We conjecture that a similar generalization may work here, thus achieving results which are in an "out-of-equilibrium" context.[5]

[5] This is pointed out also in Nyarko (1996).

6.3. *Applications*

Using the separation property and Theorems A and B we were able to (strictly) weaken the assumption of Kalai and Lehrer (1993a), i.e., absolute continuity, while maintaining the spirit of their results. We believe that this is not a unique case where absolute continuity may be relaxed. Generally speaking, whenever one utilizes the Blackwell–Dubins (1962) results, one can suspect that it is possible to weaken the absolute continuity assumption (e.g., Nyarko (1996)).

APPENDIX

In this Appendix we restate Lemmas 1, 2, and 3 and prove them

LEMMA 1. *Let* $\{p_i\}$ *and* $\{q_i\}$ *be two sequences of nonnegative numbers that sum to at most* 1. *Then,* $\sum_i |p_i - q_i| \geq d$ *implies* $\sum_i \sqrt{p_i q_i} \leq 1 - d^2/8$.

Proof. Set $p_i = a_i^2$ and $q_i = b_i^2$. Using the Cauchy–Schwartz inequality,

$$d \leq \sum |a_i^2 - b_i^2| = \sum |(a_i - b_i)(a_i + b_i)|$$
$$\leq \left(\sum (a_i - b_i)^2 \sum (a_i + b_i)^2 \right)^{1/2}. \tag{8}$$

Applying the inequality of averages,

$$\sum (a_i + b_i)^2 = \sum p_i + \sum q_i + 2 \sum \sqrt{p_i q_i}$$
$$\leq 1 + 1 + 2 \cdot \sum \frac{p_i + q_i}{2} \leq 4. \tag{9}$$

Combine (8) and (9) to get $\sum (a_i - b_i) \geq d^2/4$. This in turn implies

$$\sum \sqrt{p_i q_i} = \sum a_i b_i = \frac{\sum a_i^2 + \sum b_i^2}{2} - \frac{1}{2} \sum (a_i - b_i)^2 \leq 1 - \frac{d^2}{8}. \quad \blacksquare$$

LEMMA 2. *Suppose* μ_ε *is asymptotically* ε*-near* μ, *then* μ-a.e. *there is a time* S, s.t. $\forall s \geq S$, μ_ε *is* ε*-close to* μ.

Proof. For every integer s define the random variable Z_s as follows:

$$Z_s(\omega) = \begin{cases} 1 & \text{if } |\mu_\varepsilon(P_{s+1}(\omega) \mid P_s(\omega))/\mu(P_{s+1}(\omega) \mid P_s(\omega)) - 1| < \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of μ_ε the sequence $\{Z_s\}$ converges μ-a.s. to 1. By Lemma 1 of Kalai and Lehrer (1994), there is a time S s.t. $s \geq S$ implies that

$\mu(B_s \mid P_s(\omega)) > 1 - \varepsilon$, where $B_s = \{\omega' \in P_s(\omega); Z_s(\omega') = 1\}$. This is so for the following reason. Fix $0 < d < 1$ and define $C_m = \{\omega; |Z_s - 1| > d$ for some $s > m\}$. It is shown by Kalai and Lehrer (1994, Lemma 1) that

$$\mu\{\omega; \mu(C_m \mid P_m(\omega)) > \varepsilon \text{ for infinitely many } m\text{'s}\} = 0.$$

Therefore,

$$\mu\{\omega; \mu(B_m \mid P_m(\omega)) < 1 - \varepsilon \text{ infinitely often}\} = 0.$$

Hence, with μ probability 1 there is a random S s.t. $\mu(B_s \mid P_s(\omega)) > 1 - \varepsilon$ when $s \geq S$.

By the definition of Z_s and B_s, $|\mu_\varepsilon(B_s \mid P_s(\omega)) - \mu(B_s \mid P_s(\omega))| < \varepsilon$. Thus, $\mu_\varepsilon(B_s^c \mid P_s(\omega)) < 2\varepsilon$. Therefore for every $A \in \mathscr{P}_{s+1}$ and $A \subseteq B_s^c$ $|\mu_\varepsilon(A|P_s(\omega)) - \mu(A|P_s(\omega))| < 2\varepsilon$, as desired, while if $A \in \mathscr{P}_{s+1}$ and $A \subseteq B_s$, then, by the definition of B_s, the same inequality holds. Hence, for every $A \in \mathscr{P}_{s+1}$, $|\mu_\varepsilon(A \mid P_s(\omega))| - \mu(A \mid P_s(\omega))| < 2\varepsilon$ and the proof is complete. ■

LEMMA 3. *If $\tilde{\mu} = \alpha\mu_\varepsilon + (1 - \alpha)\lambda_\varepsilon$, where μ_ε is asymptotically ε-near μ then*

$$1 \geq \overline{\lim}\left[\frac{\tilde{\mu}(P_s(\omega))}{\mu(P_s(\omega))}\right]^{1/S} \geq \underline{\lim}\left(\frac{\tilde{\mu}(P_s(\omega))}{\mu(P_s(\omega))}\right)^{1/S} > 1 - 2\varepsilon.$$

Proof. We start with the right part of the inequality:

$$\begin{aligned}
\frac{\tilde{\mu}(P_s(\omega))}{\mu(P_s(\omega))} &= \frac{\alpha\mu_\varepsilon(P_n(\omega)) + (1 - \alpha)\lambda_\varepsilon(P_s(\omega))}{\mu(P_s(\omega))} \\
&\geq \alpha_\varepsilon \frac{\mu_\varepsilon(P_s(\omega))}{\mu(P_s(\omega))} = \alpha_\varepsilon \prod_{j=1}^{s} \frac{\mu_\varepsilon(P_j(\omega) \mid P_{j-1}(\omega))}{\mu(P_j(\omega) \mid P_{j-1}(\omega))} \\
&\geq \alpha_\varepsilon \prod_{j=1}^{N(\varepsilon,\omega)} \frac{\mu_\varepsilon(P_j(\omega) \mid P_{j-1}(\omega))}{\mu(P_j(\omega) \mid P_{j-1}(\omega))} \prod_{j=N(\varepsilon,\omega)+1}^{s} \frac{\mu_\varepsilon(P_j(\omega) \mid P_{j-1}(\omega))}{\mu(P_j(\omega) \mid P_{j-1}(\omega))} \\
&\geq \alpha_\varepsilon \prod_{j=1}^{N(\varepsilon,\omega)} \frac{\mu_\varepsilon(P_j(\omega) \mid P_{j-1}(\omega))}{\mu(P_j(\omega) \mid P_{j-1}(\omega))} (1 - 2\varepsilon)^{s-N(\varepsilon,\omega)}.
\end{aligned}$$

So,

$$\liminf\left(\frac{\tilde{\mu}(P_s(\omega))}{\mu(P_s(\omega))}\right)^{1/s} \geq 1 - 2\varepsilon.$$

The left part of the inequality is exactly the claim of Lemma 1 in Lehrer and Smorodinsky (1996) and thus we are finished. ■

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