A Cross Entropy Based Multi-Agent Approach to Traffic Assignment Problems

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Summary. In this paper, we propose a Cross Entropy (CE) [1] based multiagent approach for solving static/dynamic traffic assignment problems (TAP). This algorithm utilizes a family of probability distributions in order to guide travelers (agents) to network equilibrium. The route choice probability distribution depends on the average network performance experienced by agents on previous days. Based on the minimization of cross entropy concept, optimal probability distributions are derived iteratively such that high quality routes are more attractive to agents. The advantage of the CE method is that it is based on a mathematical framework and sampling theory, in order to derive the optimal probability distributions guiding agents to the dynamic system equilibrium. Interestingly, we demonstrate that the proposed approach based on CE method coincides with dynamic system approaches. Numerical studies illustrate both nonlinear and bimodal static traffic assignment problems. A comparative study of the proposed method and the dynamic system approach is provided to justify the efficiency of proposed method.

1 Introduction

Solving both static/dynamic traffic assignment problems based on realistic traffic flow models is an important issue in transportation research. Following the Wardrop equilibrium principle, drivers are assumed to swap to cheaper cost routes until the costs of all routes actually used are equal and are less than the costs of unused routes. The traffic assignment problems can be usually formulated as variational inequality, non-linear complementarity or fixed point problems [2]. For one user class static traffic assignment problems, route travel cost is evaluated based on a performance function under monotone, symmetric or separable assumptions. Based on these assumptions, classical convex optimization techniques can be applied for solving these problems without difficulty. However, for dynamic or multimodal traffic assignment problems, the classical solution methods can be no more applicable, given that the analytical form of link cost function is generally difficult to derive and also that the monotonicity assumption is no longer respected in the case of multimodal network [3].

To address this issue, there exist many variant approaches in the literature. We are particularly interested in dynamical system approaches [4–6] for comparative study with the proposed CE based method. The dynamical system approaches are based on a day-to-day route flow adjustment process, for which network flows between the same OD pair are adjusted following some deterministic or stochastic procedure until (Wardrop) user equilibrium is achieved. However, the adjustment process is problematic, especially when there exist more than one class flows between each OD pair. More explicitly, there is no systematical way to adjust all class flows with consideration of the asymmetric effect caused by other class users on the network. Another issue is that the dynamical system approach cannot be applied when there are multiple equilibrium in multimodal network. Indeed, there is no Lyapunov function in this case. On the other hand, the dynamical system approach could be applied to each mode in sequence.

In this work, we derive iteratively the optimal route choice probability distributions towards network equilibrium based on the CE Method, which is a stochastic optimization technique for combinatorial and continuous optimization problems. Based on the minimization of cross entropy concept, optimal probability distributions are derived iteratively such that high quality routes are more attractive to agents. We demonstrate that the proposed CE method can be seen as a stochastic version of the dynamical system approach, which is based on importance sampling and rare event theory. Two numerical examples are illustrated for one and multiple user class static traffic assignment problems. A comparaison study of the proposed method with dynamical system approach is also shown in order to evaluate the efficiency of proposed method.

2 The Cross Entropy Based Multi-Agent Method

The CE method is an adaptive algorithm for estimating probabilities of rare events in complex systems. Based on this method, we can modelize user equilibria as rare events depending on complex stochastic behavior of all travellers. Consider a static traffic assignment or network equilibrium problem, for which a network consists in an origin-destination (OD) pair connected by a set of routes. Agents are located at origins and choose a route stochastically to a destination based on a route choice probability distribution. Note that the probability distribution is related to each OD pair and utilized by agents of the same OD pair. Let C_i denote travel cost of route i. For simplicity of the exposition we assume here that C_i is a function of its demand, i.e. $C_i = C_i(d_i), \forall i \in I$; and I denotes the set of all routes connecting the same OD pair. Let p_i denote the choice probability of route i and $H_i(\gamma)$

denote the performance function of route i, defined by Boltzmann distribution [7],

$$H_i(\gamma) = e^{-C_i(d_i)/\gamma}, \quad \forall i \in I, \tag{1}$$

where γ is the control parameter or temperature. We note that γ controls the swapping force pushing flow to shift from more expensive routes to cheaper ones. Note that as γ decreases, the swapping force increase. The overall expected performance based on the choice probability distribution $\mathbf{p}$ is the expectation of all route performance, defined as:

$$h(\mathbf{p}, \gamma) = \sum_{i \in I} p_i H_i(\gamma), \tag{2}$$

subject to:

$$\sum_{i \in I} p_i = 1, \quad \forall p_i \geq 0 \tag{3}$$

As mentioned above, agents swap from costly routes to cheaper ones in order to minimize their own cost. Following Rubinstein [1], the optimal choice probability distribution towards to cheaper routes at next iteration is the solution of the following optimization problem which minimize the Kullback-Liebler relative entropy between two consecutive probability distributions,

$$\mathbf{p}^{w+1} = \max_{\mathbf{p}} \mathbb{E}_{\mathbf{p}^w}[H(\gamma) \ln \mathbf{p}], \tag{4}$$

subject to equation (3), where w is the index of iteration.

The optimization problem of equation (4) is equivalent to

$$\mathbf{p}^{w+1} = \arg\min_{\mathbf{p}} \sum_{i \in I} p_i^w e^{\frac{C_i(d_i)}{\gamma}} \ln p_i, \tag{5}$$

subject to equation (3).

We can derive the solution of the above equation as:

$$p_i^{w+1} = p_i^w \times \frac{e^{-C_i^w/\gamma^w}}{\sum_{j \in I} p_j^w e^{-C_j^w/\gamma^w}}, \quad \forall i \in I \tag{6}$$

We note that the derived probability distribution depends on the control parameter γ, which must be estimated adaptively in order to converge to the user equilibrium.

As mentioned above, keep γ small will cause more agents swapping to routes which are cheap. However, swapping flow too early will cause traffic congestion on the chosen routes and make them unattractive since the cost-flow relationship is nonlinear. Thus keeping γ too small may renforce routes that turn out to be inefficient. An adaptive parameter estimation technique to obtain optimal γ is to minimize γ, provided that the sum over the differences between two consecutive probability distributions is bounded. This bound is decreased iteratively towards 0 like the cooling technique utilized in simulated annealing method [8]. Thus, we consider the following adaptive estimation of the control parameter γ:

$$\min \gamma^w \text{s.t.} \sum_{i \in I} |p_i^{w+1} - p_i^w| \leq \alpha^w, \tag{7}$$

where p_i^w is defined by equation (5), $\alpha^w = \frac{C}{w}$ a numerical divergent series, the generic term of which converges to 0, and C a constant for setting initial value of γ.

If the first order approximation is applied to exponential function, i.e.

$$e^{-C_i^w/\gamma^w} \simeq 1 - \frac{C_i^w}{\gamma^w}, \quad \text{if } \gamma^w \to \infty \tag{8}$$

We can obtain the following dynamical system which is the same as the system introduced in [6]:

$$\dot{\mathbf{p}} = -\mathbf{p}(C(\mathbf{p}) - \overline{C}(\mathbf{p})), \tag{9}$$

where $\overline{C}(\mathbf{p}) = \mathbb{E}_{\mathbf{p}} C(\mathbf{p})$ denotes the expectation of travel cost with respect to the probability distribution $\mathbf{p}$.

We note that at Wardrop equilibrium, the derived probability distribution is stable. i.e.

$$p_i^{w+1} = p_i^w, \quad \text{if } C_i^w = \min_{j \in I} C_j^w, \tag{10}$$

which coincides with the result obtained by the dynamical system approach. Note that the classical CE method has been tailored for cost functions independent of demand, which is why it diverges from the adaptation presented here.

Finally, let us investigate its impact on the derived probability changes of γ tending to ∞ and 0, respectively.

In the case of $\gamma^w \to \infty$, we obtain the following results

$$\lim_{\gamma^w \to \infty} (p_i^{w+1} - p_i^w) = 0, \quad \forall w \tag{11}$$

On the contrary, when $\gamma^w \to 0$, we obtain

$$\lim_{\gamma^w \to 0} (p_i^{w+1} - p_i^w) = -p_i^w \left(1 - \frac{1}{\sum_{j \in I} p_j^w e^{\frac{C_i^w - C_j^w}{\gamma^w}}} \right) = \begin{cases} -p_i^w & \text{if } C_i^w > \min_{j \in I} C_j^w \\ -\infty & \text{otherwise} \end{cases} \tag{12}$$

3 The Algorithm Based on CE Method

The algorithm for solving general TAPs consists of an iterative procedure which updates adaptively the probability distributions and the control parameters until the desired result is obtained. The algorithm is described as follows:

Main algorithm

1. *Initialization*
   a) Initialize the parameter α^w according to the magnitude of route choice set and set iteration index $w = 0$.
   b) Initialize route choice probability according to a uniform probability $p_i^w = 1/|I_k|, \forall i \in I_k, \forall k \in K$, where I_k is the set of routes connecting the same O/D pair k, K is the set of all OD pairs.
2. *Compute travel cost*
   Compute travel costs of all routes with respect to agents' choice. In the static case, travel cost can be evaluated by cost-flow functions. However, in the dynamic case, agents' experienced travel costs can be computed with respect to analytical or simulation-based traffic flow models. Different traffic flow models can be utilized, such as first-order model [9, 10], the generic second-order traffic flow model [11, 12] or the cell transition model [13]. Moreover, different alternatives with related cost specifications can be taken into account, e.g. mode choice, departure time choice or destination choice etc.
3. *Update the choice probability distribution*
   For adaptive estimation of γ, we utilize the interpolation method by bounding $0 \leq \gamma \leq M$, where M is a constant. The optimal γ can be deduced from equation 7. The derived optimal probability distribution can be thus evaluated by equation 6.
4. *Stopping criteria*
   When $w = w_{max}$ or the derived choice probability distributions are stable, stop; otherwise go to step 2.

We note that for multimodal traffic assignment problems, there is one user equilibrium per class of users, given the other classes of users' choice. Accordingly, the probability distributions utilized by agents of different classes should be distinguished.

4 Numerical Studies

In this section, we study two static traffic assignment problems with respect to one and multiple classes. The first example is shown in Fig. 1 consisting of an OD pair with three links [6]. The nonlinear cost-flow functions are the following:

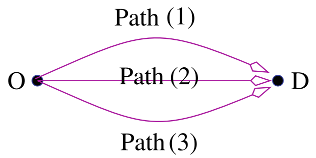

Fig. 1. A static mono-modal network.

$$\begin{aligned} t_1 &= 10(1.0 + 0.15(\frac{x_1}{2})^4) \\ t_2 &= 20(1.0 + 0.15(\frac{x_2}{4})^4) \\ t_3 &= 25(1.0 + 0.15(\frac{x_3}{3})^4) \end{aligned} \tag{13}$$

$$x_1 + x_2 + x_3 = 10$$

where x_i is the route flow on link/path i and the travel demand is 10. We initialize the probability distribution for route (link) choice as uniform. The evolution of the probability distribution is shown in Fig. 2. We see that after 15 iterations, the probability distribution for link choice becomes stable, and the link cost over all links is almost equal, as shown in Fig. 3. The estimates of γ oscillate after the solution nears the network equilibrium, as shown in Fig. 4.

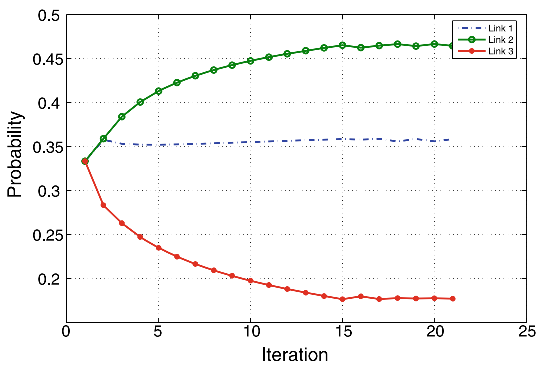

Fig. 2. Evolution of link choice probability.

The second example concerns a bimodal traffic assignment problem and is described in Fig. 5. Two classes of users, e.g. car and bus, are present in a two link/path network. The linear flow-cost function is given by [3]:

$$\begin{aligned} t_{ai} &= 1.5x_{ai} + 5x_{bi} + 30, \forall i \in \{1, 2\} \\ t_{bi} &= 1.3x_{ai} + 2.6x_{bi} + 28, \forall i \in \{1, 2\} \end{aligned} \tag{14}$$

$$d_a = 16, d_b = 4$$

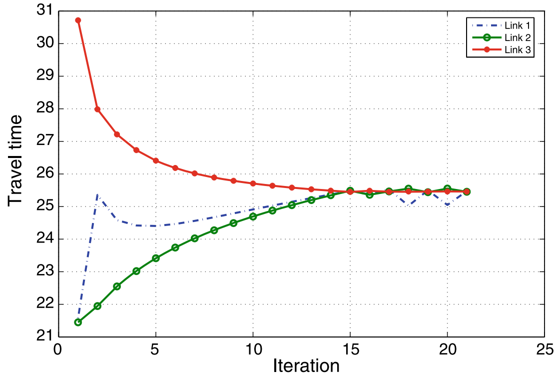

Fig. 3. Evolution of link travel time.

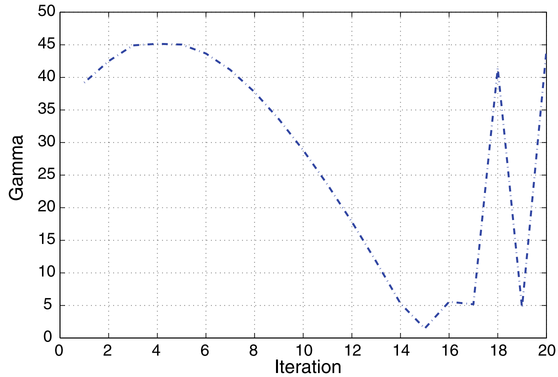

Fig. 4. Evolution of control parameter γ.

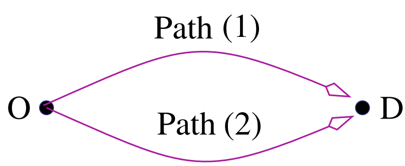

Fig. 5. A static bimodal network.

where t_{ki} is the travel time of class k on link i, x_{ki} is the number of class k on link i. d_k is the demand of class k. To treat this example, we introduce two probability distributions with respect to the two user classes. Accordingly, two different control parameters γ need to be estimated. We initialize the probability distribution as a uniform distribution for each class. Similarly to the previous example, we obtain convergence towards a user equilibrium. The Nagurney field has as components the differences between path costs per mode on link (1) and (2) as functions of the modal flows on link (1). The Nagurney field is shown in Fig. 6, where the equilibrium points are represented by red stars. The flow vector far from equilibrium converges to the stable equilibria at the up-left corner or down-right corner according to different initial conditions. The unstable equilibrium can be found at the center with flows being 8 and 2 for class a and b.

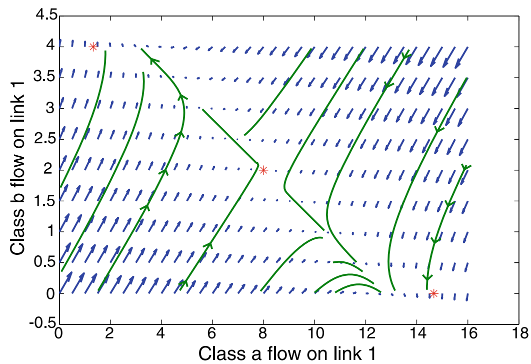

Fig. 6. The velocity vectors associated to the Nagurney field on link 1.

5 Comparison with Dynamical System Approach

In this section, we are interested in comparing our method with other approaches. Note that the projection-type method [5] attempts iteratively to a find a fixed point but its convergence speed is slow (linear or less). For this reason, we choose the dynamical system approach to compare our proposed CE method. The network is composed of 8 nodes and 12 links with two origins and two destinations, as shown in Fig. 7. The OD travel demand equals to 1000 for each OD. The nonlinear cost functions are defined as follows:

$$
\begin{aligned}
t_1 &= 20(1.0+0.15(\frac{x_1}{4})^4) & t_7 &= 10(1.0+0.15(\frac{x_7}{2})^4) \\
t_2 &= 15(1.0+0.15(\frac{x_2}{4})^4) & t_8 &= 20(1.0+0.15(\frac{x_8}{4})^4) \\
t_3 &= 25(1.0+0.15(\frac{x_3}{3})^4) & t_9 &= 25(1.0+0.15(\frac{x_9}{3})^4) \\
t_4 &= 10(1.0+0.15(\frac{x_4}{2})^4) & t_{10} &= 90(1.0+0.15(\frac{x_{10}}{2})^4) \\
t_5 &= 20(1.0+0.15(\frac{x_5}{4})^4) & t_{11} &= 20(1.0+0.15(\frac{x_{11}}{4})^4) \\
t_6 &= 5(1.0+0.15(\frac{x_6}{3})^4) & t_{12} &= 35(1.0+0.15(\frac{x_{12}}{3})^4)
\end{aligned}
\qquad (15)
$$

$$d_k = 1000, \forall k \in K,$$

where x_i denotes number of travelers, k an OD pair. The network equilibrium solutions obtained by these two approaches (CE and dynamic system) are very close for all OD pairs. The evolution of path costs, connecting the same OD pair (O1-D1), obtained by these two methods is shown in Fig. 8. The CE method converges to equilibrium faster than the dynamical system approach.

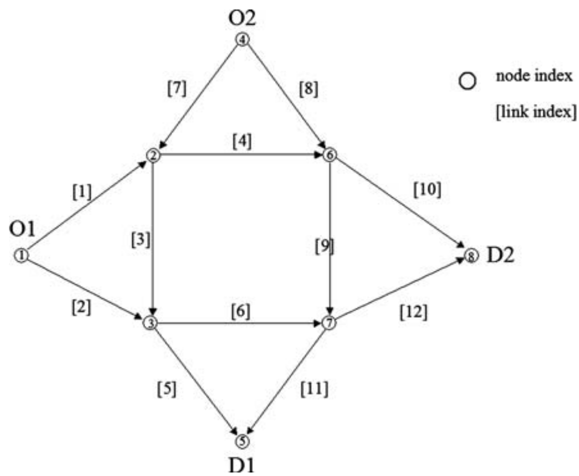

Fig. 7. The multiple OD network.

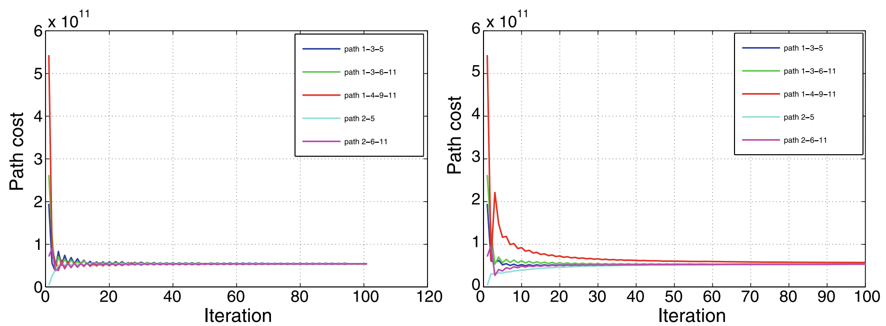

Fig. 8. Evolution of path costs connecting origin O1 and destination D1 based on CE method (left) and dynamical system approach [4] (right).

6 Conclusion

In this study, a fast convergent CE based multi-agent approach for solving the general static/dynamic traffic assignment problem has been proposed. Based on the CE method, the adaptive route flow shifts are derived, resulting from travel cost realized on previous day. We have also shown that in a multimodal network, given that the monotonicity of the travel cost function does not hold, multiple user equilibrium solutions can be found quickly with different initial conditions. The results with comparison to the dynamical system approach show that the proposed CE method approach converges more quickly. It is also better adapted to complex cost functions, as in the dynamic assignment problems based on traffic flow models.

We have provided the main algorithm for solving general static/dynamic traffic assignment problems. Further applications in large scale multimodal network based on realistic traffic flow models are actually under study.

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